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Duality for finite Hilbert algebras

Sergio Celani, Leonardo Cabrer

CONICET and Departamento de Matemáticas, Universidad Nacional del Centro, Pinto 399, 7000 Tandil, Argentina

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Abstract

In this work we shall give a characterization of the Hilbert algebras given by the order and we will prove a duality for finite Hilbert algebras by means of finite ordered sets endowed with a distinguished set of subsets. We will also study the case when the finite Hilbert algebras are join-semilattices or meet-semilattices relative to the natural order defined by the implication. Finally we will prove that Hilbert do not admit a natural duality.

Keywords: Hilbert algebras; Ordered sets; Irreducible elements; **H**-spaces; Meet-semilattices; Join-semilattices; Brouwerian semilattices

1. Introduction and preliminaries

Hilbert algebras represent the algebraic counterpart of the implicative fragment of Intuitionistic Propositional Logic. In [5] Diego gives a topological representation for Hilbert algebras and he proves that every Hilbert algebra is isomorphic to a subalgebra of the implicative reduct of a Heyting algebra generated by a certain topological space. This result is a generalization of the known results given by M. Stone for Heyting algebras. Other representation theorem can be given by using poset. In [2] the first author of this paper proves that for every Hilbert $\mathbf{A}$ algebra there exists a poset $\langle X, \leqslant\rangle$ such that $\mathbf{A}$ is isomorphic to a subalgebra of the implicative reduct of the Heyting algebra $\mathscr{P}_i(X)$ of increasing subsets of $\langle X, \leqslant\rangle$. Unfortunately, these representations do not give a full duality. However, using some results of [2] and some ideas of [3], it is possible to give a duality for the case of

E-mail addresses: scelani@exa.unicen.edu.ar (S. Celani), lcabrer@exa.unicen.edu.ar (L. Cabrer).

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finite Hilbert algebras. *This is the main objective of the present paper*. We shall prove that the dual space of a finite Hilbert algebra $\mathbf{A}$ is a structure $\langle X, \leqslant, \mathcal{S}\rangle$, called *Hilbert space*, where $\langle X, \leqslant\rangle$ is a poset and $\mathcal{S}$ is a non-empty subset of the power set $\mathcal{P}(X)$. This duality is a full duality, i.e., the category of finite Hilbert algebras with homomorphisms is dually equivalent to the category of finite Hilbert spaces with adequate morphisms.

In the remaining part of this section we shall review some results on Hilbert algebras and we shall introduce notations and some new definitions. In Section 2 we shall define the notion of Hilbert algebra given by the order and we will prove a characterization of this class of Hilbert algebras. In Section 3 we will give the above mentioned representation and duality for finite Hilbert algebras by means of the so-called **H**-spaces. In Section 4 we will introduce an important reflective subcategory of **H**-spaces. In Section 5 we shall study the case of finite Hilbert algebra with lattice operations, i.e., finite Hilbert algebras where the natural order defines a meet or join operation. In Section 6 we will prove that Hilbert algebras do not admit a natural duality, proving that the variety of Hilbert algebras is not generated by any finite algebra.

For the rest of this paper we will use the following conventions. Given a set X and a subset $Y \subseteq X$ we will note $X\backslash Y = \{x \in X : x \notin Y\}$, when there is no way to confusion we will note $X\backslash Y = Y^c$. Given $R \subseteq X \times Y$, we will note $R(x) = \{z \in Y : (x, z) \in R\}$ and $R^{-1}(y) = \{z \in X : (z, y) \in R\}$.

Given a poset $\langle X, \leqslant\rangle$, a set $Y \subseteq X$ is called *increasing* if it is closed under $\leqslant$, i.e., if for every $x \in Y$ and every $y \in X$, if $x \leqslant y$ then $y \in Y$. Dually, $Y \subseteq X$ is said to be *decreasing* if for every $x \in Y$ and every $y \in X$, if $y \leqslant x$ then $y \in Y$. We will note $= \{x \in X : y \leqslant x$ for some $y \in Y\}$ to the least increasing set that contain Y and analogously $(Y] = \{x \in X : x \leqslant y$ for some $y \in Y\}$ to the least decreasing set that contain Y. When $Y = \{x\}$, we will note $[Y)$ $((Y])$ by $[x)$ $((x])$. The set of all increasing subsets of X will be denoted by $\mathcal{P}_i(X)$, and the power set of X by $\mathcal{P}(X)$.

Definition 1. A Hilbert algebra is an algebra $\mathbf{A} = \langle A, \rightarrow, 1\rangle$ of type $(2, 0)$ such that the following axioms hold in $\mathbf{A}$:

H1. $a \rightarrow (b \rightarrow a) = 1$.
H2. $(a \rightarrow (b \rightarrow c)) \rightarrow ((a \rightarrow b) \rightarrow (a \rightarrow c)) = 1$.
H3. $a \rightarrow b = 1 = b \rightarrow a$ imply $a = b$.

$\mathbf{A}$ is called trivial if $A = \{1\}$.

In [5] Diego proves that the class of Hilbert algebras form a variety which is denoted by $\mathbb{H}$.

In this paper the symbol $\Rightarrow$ is used for logical implication, and $\Leftrightarrow$ for logical equivalence.

Lemma 2. *Let* $\mathbf{A} = \langle A, \rightarrow, 1\rangle \in \mathbb{H}$. *Then for every* $a, b, c \in A$ *the following assertions are satisfied*:

1. $a \rightarrow a = 1$.
2. $1 \rightarrow a = a$.

3. $a \to (b \to c) = b \to (a \to c)$.
4. $a \to (b \to c) = (a \to b) \to (a \to c)$.
5. $a \to ((a \to b) \to b) = 1$.
6. $a \to (a \to b) = a \to b$.
7. $((a \to b) \to b) \to b = a \to b$.
8. $(a \to b) \to ((b \to a) \to a) = (b \to a) \to ((a \to b) \to b)$.

It is easy to see that the binary relation $\leqslant$ defined in a Hilbert algebra $\mathbf{A}$ by

$$a \leqslant b \quad \text{if and only if} \quad a \to b = 1,$$

is a partial order on A with greatest element 1. This order is called the *natural ordering* on A.

Example 3. Let $\langle X, \leqslant \rangle$ be a poset. It is easy to see that $\langle \mathscr{P}_i(X), \longmapsto, X \rangle$ is a Hilbert algebra where the implication $\longmapsto$ is defined by

$$U \longmapsto V = \{x \in X : [x) \cap U \subseteq V\},$$

for $U, V \in \mathscr{P}_i(X)$. We can see that in the definition of $\longmapsto$ is not necessary that U, $V \in \mathscr{P}_i(X)$, since $U \longmapsto V \in \mathscr{P}_i(X)$, for every $U, V \in \mathscr{P}(X)$.

Remark 4. Note that if $\langle X, \leqslant \rangle$ is a poset and $x, y \in X$, then

$$x \leqslant y \Leftrightarrow \{y\} \longmapsto \emptyset \subseteq \{x\} \longmapsto \emptyset \Leftrightarrow (\{x\} \longmapsto \emptyset]^c \subseteq (\{y\} \longmapsto \emptyset]^c].$$

Given $\mathbf{A} \in \mathbb{H}$ and $a, a_1, \ldots, a_n \in A$, we define:

$$(a_n, \ldots, a_1; a) = \begin{cases} a_1 \to a & \text{if } n = 1, \\ a_n \to (a_{n-1}, \ldots, a_1; a) & \text{if } n > 1. \end{cases}$$

Let $\mathbf{A} \in \mathbb{H}$. A subset $D \subseteq A$ is a *deductive system* of $\mathbf{A}$ if $1 \in D$, and if $a, a \to b \in D$ then $b \in D$. The set of all deductive systems of a Hilbert algebra $\mathbf{A}$ is noted $D_{\mathrm{s}}(\mathbf{A})$. It is easy to prove that $D_{\mathrm{s}}(\mathbf{A})$ is closed under arbitrary intersections. So, given $X \subseteq A$ the set $\langle X \rangle = \bigcap \{D \in D_{\mathrm{s}}(\mathbf{A}) : X \subseteq D\}$ is called the deductive system generated by X. If $X = \{a\}$, then we will denote $\langle X \rangle = \langle a \rangle$. Let $D \in D_{\mathrm{s}}(\mathbf{A})$. If there exists some $a \in A$ such that $D = \langle a \rangle$ we shall say that D is principal. The deductive system generated by a subset $X \subseteq A$ can be characterized as the set:

$$\langle X \rangle = \{a \in A : \text{there exist } a_1, \ldots, a_n \in X \text{ such that } (a_1, \ldots, a_n; a) = 1\}.$$

In consequence, we get that $\langle a \rangle = \{b \in A : a \leqslant b\} = [a)$.

Let $\mathbf{A} \in \mathbb{H}$. Let $D \in D_{\mathrm{s}}(\mathbf{A}) - \{A\}$. We shall say that D is *irreducible* if and only if for any $D_1, D_2 \in D_{\mathrm{s}}(\mathbf{A})$ such that $D = D_1 \cap D_2$, it follows that $D = D_1$ or $D = D_2$. We shall say that D is *completely irreducible* if and only if for any family $\{D_i : i \in I\} \subseteq D_{\mathrm{s}}(A)$ such that $D = \bigcap \{D_i : i \in I\}$, then $D = D_i$ for some $i \in I$. The set of all irreducible (completely irreducible) deductive systems of a Hilbert Algebra $\mathbf{A}$ is noted $D_{\mathrm{si}}(\mathbf{A})$ ($D_{\mathrm{ci}}(\mathbf{A})$). It is clear that $D_{\mathrm{ci}}(\mathbf{A}) \subseteq D_{\mathrm{si}}(\mathbf{A})$. If $\mathbf{A}$ is a finite Hilbert algebra, then $D_{\mathrm{ci}}(\mathbf{A}) = D_{\mathrm{si}}(\mathbf{A})$. For a proof of the next theorems see [5] or [1] or [9].

Theorem 5. *Let* $\mathbf{A} \in \mathbb{H}$. *Let* $D \in D_{\mathrm{s}}(\mathbf{A}) - \{A\}$. *Then the following conditions are equivalent*:

1. $D \in D_{\mathrm{si}}(\mathbf{A})$.
2. *If* $a, b \notin D$ *there exists* $c \notin D$ *such that* $a, b \leqslant c$.
3. *If* $a, b \notin D$ *there exists* $c \notin D$ *such that* $a \to c, b \to c \in D$.

Theorem 6. *Let* $\mathbf{A} \in \mathbb{H}$. *Let* $D \in D_{\mathrm{s}}(\mathbf{A})$. *Then the following conditions are equivalent*:

1. $D \in D_{\mathrm{ci}}(\mathbf{A})$.
2. *There exists* $a \notin D$ *such that if* $D \subsetneq E \in D_{\mathrm{s}}(A)$, *then* $a \in E$.
3. *There exists* $a \notin D$ *such that* $b \to a \in D$, *for every* $b \notin D$.

Let $D \in D_{\mathrm{s}}(\mathbf{A})$ and $a \in A$. We shall say that a is *associated with* D or that D is *maximal relative to* a, if D is maximal with respect to the deductive systems which do not contain a, i.e., D is maximal in the set $\{E \in D_{\mathrm{s}}(\mathbf{A}) : a \notin E\}$.

From the above theorem we can deduce the next corollary:

Corollary 7. *Let* $\mathbf{A} \in \mathbb{H}$. *Then the following conditions are satisfied*:

1. *If* $D \in D_{\mathrm{s}}(\mathbf{A})$ *and* $a \notin D$ *then there exists* $Q \in D_{\mathrm{ci}}(\mathbf{A})$ *such that* $a \notin Q$ (*moreover*, *a is associated with* Q) *and* $D \subseteq Q$.
2. *If* $a, b \in A$ *are such that* $a \nleqslant b$ *then there exists* $Q \in D_{\mathrm{ci}}(\mathbf{A})$ *such that* $b \notin Q$ (*moreover* *b is associated with* Q) *and* $a \in Q$.

Note that the previous corollary proves that for every $D \in D_{\mathrm{s}}(\mathbf{A})$ we have that

$$D = \bigcap_{\substack{D \subseteq Q \\ Q \in D_{\mathrm{ci}}(\mathbf{A})}} Q$$

Definition 8. Let $\mathbf{A} \in \mathbb{H}$. We define a relation $K_{\mathbf{A}} \subseteq D_{\mathrm{ci}}(\mathbf{A}) \times A$ as follows:

$$(D, a) \in K_{\mathbf{A}} \text{ if and only if } a \text{ is associated with } D.$$

When there is no risk of ambiguity we will omit the subscript and note K instead of $K_{\mathbf{A}}$.

Lemma 9. *Let* $\mathbf{A} \in \mathbb{H}$. *Let* $D \in D_{\mathrm{ci}}(\mathbf{A})$. *For each* $a, b \in K(D)$ *there exists* $c \in A$ *such that* $a \leqslant c$, $b \leqslant c$ *and* $c \in K(D)$.

Proof. Let $a, b \in K(D)$. Since $a, b \notin D$ and D is also irreducible, by Theorem 5 we get that there exists $c \notin D$ such that $a \leqslant c$ and $b \leqslant c$. There exists a deductive system E associated with c such that $D \subseteq E$, then $a \notin E$. Then $E = D$ because D is maximal relative to a. Thus $c \in K(D)$. $\square$

Let $\mathbf{A}$ be Hilbert algebra. Let us consider the poset $\langle D_{\mathrm{ci}}(\mathbf{A}), \subseteq \rangle$ and the mapping

$$\varphi_{\mathbf{A}} : \mathbf{A} \to \mathscr{P}_i(D_{\mathrm{ci}}(\mathbf{A}))$$

defined by

$$\varphi_{\mathbf{A}}(a) = \{P \in D_{\mathrm{ci}}(\mathbf{A}) : a \in P\}.$$

For a proof of the next theorem see [5] or [2].

Theorem 10. *Let* $\mathbf{A} \in \mathbb{H}$. *Then* $\mathbf{A}$ *is isomorphic to the subalgebra* $\varphi_{\mathbf{A}}(\mathbf{A}) = \{\varphi_{\mathbf{A}}(a) : a \in A\}$ *of* $\langle \mathscr{P}_i(D_{\mathrm{ci}}(\mathbf{A})), \longmapsto, D_{\mathrm{ci}}(\mathbf{A})\rangle$.

2. Hilbert algebras given by the order

In [5] Diego introduce the Hilbert algebras given by order. This kind of Hilbert algebras will be useful later to construct examples. In this section we will characterize the Hilbert algebras given by order by means of its deductive systems and by means of its irreducible elements.

Definition 11. A Hilbert algebra $\mathbf{A}$ is called *given by the order* $\leqslant$ if and only if for every $a, b \in A$,

$$a \to b = \begin{cases} b & \text{if } a \nleqslant b, \\ 1 & \text{if } a \leqslant b. \end{cases}$$

Definition 12. Let $\mathbf{A} \in \mathbb{H}$. An element $p \in A - \{1\}$ is called *irreducible* if for each $a \in A$, $a \leqslant p$ or $a \to p = p$.

In the following result we shall give an useful characterization of irreducible elements.

Lemma 13. *Let* $\mathbf{A} \in \mathbb{H}$. *Let* $p \in A$. *Then the following conditions are equivalent*:

1. $p \neq 1$ *and for every finite subset* $\{a_1, a_2, \ldots, a_n\} \subseteq A$, $(a_1, \ldots, a_n; p) \neq 1$ *when* $a_i \nleqslant p$, *for any* $1 \leqslant i \leqslant n$.
2. $(p]^c \in D_{\mathrm{s}}(\mathbf{A})$.
3. p *is irreducible*.
4. *For every* $a, b \in A$, *if* $p \in \langle\{a, b\}\rangle$, *then* $a \leqslant p$ *or* $b \leqslant p$.

Proof. $1 \Rightarrow 2$. We will prove that $\langle (p]^c \rangle = (p]^c$.

Obviously $(p]^c \subseteq \langle (p]^c \rangle$. If $b \in \langle (p]^c \rangle$, there exist $\{a_1, a_2, \ldots, a_n\} \subseteq (p]^c$ such that $(a_1, \ldots, a_n; b) = 1$. Suppose that $b \in (p]$, i.e., $b \to p = 1$. Then

$$\begin{aligned} 1 &= (a_1, \ldots, a_n; 1) = (a_1, \ldots, a_n; b \to p) \\ &= (a_1, \ldots, a_n; b) \to (a_1, \ldots, a_n; p) \\ &= (a_1, \ldots, a_n; p), \end{aligned}$$

which is a contradiction. So, $b \in (p]^c$. Therefore $(p]^c \in D_{\mathrm{s}}(\mathbf{A})$.

$2 \Rightarrow 3$. Let $a \in A$ and let us suppose that $a \nleqslant p$. Then $a \in (p]^c$. If $a \to p \nleqslant p$, $a \to p \in (p]^c$. But since $(p]^c \in D_{\mathrm{s}}(\mathbf{A})$, $p \in (p]^c$, which is a contradiction. So $a \to p \leqslant p$, and thus $a \to p = p$.

$3 \Rightarrow 4$. Let $a, b \in A$ such that $p \in \langle\{a, b\}\rangle$. Then $a \to (b \to p) = 1$. Let us suppose that $a \nleqslant p$. Since p is irreducible, $a \to p = p$. Thus, $1 = b \to (a \to p) = b \to p$, i.e., $b \leqslant p$.

$4 \Rightarrow 3$. Let $a \in A$ and let us suppose that $a \nleqslant p$. By H1, we have that $p \to (a \to p) = 1$. Clearly $p \in \langle\{a, a \to p\}\rangle$. By hypothesis, $a \to p \leqslant p$, i.e., $(a \to p) \to p = 1$. By H3, $a \to p = p$. Thus p is irreducible.

$3 \Rightarrow 1$. It is easy and left to the reader. □

Remark 14. Let $\mathbf{A} \in \mathbb{H}$, and $p \in A$. It is easy to see that if $(p]^c \in D_s(\mathbf{A})$, then $(p]^c \in D_{ci}(\mathbf{A})$, since for every $D \in D_s(\mathbf{A})$, if $p \notin D$ then $D \subseteq (p]^c$.

In the next result we will characterize the Hilbert algebras given by the order.

Theorem 15. *Let* $\mathbf{A} \in \mathbb{H}$. *Then the following conditions are equivalent*:

1. $\mathbf{A}$ *is a Hilbert algebra given by the order*.
2. $\mathscr{P}_i(A) - \{\emptyset\} = D_s(\mathbf{A})$.
3. *If* $p \in A - \{1\}$, *then* $(p]^c \in D_{ci}(\mathbf{A})$.
4. *For every* $p \in A - \{1\}$, $K^{-1}(p) = \{D\}$ ($K^{-1}(p)$ *has only one element*).

Proof. $1 \Rightarrow 2$. We already know that $D_s(\mathbf{A}) \subseteq \mathscr{P}_i(A) - \{\emptyset\}$. Let $D \in \mathscr{P}_i(A) - \{\emptyset\}$. Obviously $1 \in D$. Let $a, a \to b \in D$. If $a \leqslant b$, then $b \in D$, because D is increasing. If $a \nleqslant b$ then $a \to b = b$. So, $b \in D$. Thus, $D \in D_s(\mathbf{A})$.

$2 \Rightarrow 1$. Let $a, b \in A$. If $a \leqslant b$, then $a \to b = 1$. If $a \nleqslant b$, then $b \notin \langle a\rangle$. Let us consider the set $D = \langle a\rangle \cup \langle a \to b\rangle$. Clearly $D \in \mathscr{P}_i(A) - \{\emptyset\} = D_s(\mathbf{A})$. Thus $D \in D_s(\mathbf{A})$. Then $b \in D$, but since $b \notin \langle a\rangle$, $b \in \langle a \to b\rangle$. It follows that $(a \to b) \to b = 1$. By H1, we have that $b \to (a \to b) = 1$, and from H3 we can deduce that $b = a \to b$. Thus $\mathbf{A}$ is a Hilbert algebra given by the order.

$2 \Rightarrow 3$. By hypothesis, if $p \neq 1$, we have that $(p]^c \in D_s(\mathbf{A})$, because $\emptyset \neq (p]^c \in \mathscr{P}_i(A)$. Clearly p is associated with $(p]^c$. From Theorem 6 and Remark 14 we get $(p]^c \in D_{ci}(A)$.

$3 \Rightarrow 4$. Let $p \in A - \{1\}$. By hypothesis $(p]^c \in K^{-1}(p)$. If $D \in K^{-1}(p)$, then $p \notin D$ and $(p] \cap D = \emptyset$, because D is increasing. Thus $D \subseteq (p]^c$. By Theorem 6 we have $D = (p]^c$. Thus, $K^{-1}(p) = \{(p]^c\}$.

$4 \Rightarrow 2$. Suppose that $K^{-1}(p) = \{D\}$ for every $p \in A - \{1\}$. Let $E \in \mathscr{P}_i(A) - \{\emptyset\}$.

Let $p \notin E$. For each $a \in E$, $a \nleqslant p$. So, by Corollary 7 there exists $Q_p \in K^{-1}(p)$ such $a \in Q_p$. As $K^{-1}(p) = \{Q_p\}$, $E \subseteq Q_p$ for each $p \notin E$. We conclude that

$$E = \bigcap_{p \notin E} Q_p \in D_s(\mathbf{A}).$$

Therefore, $\mathscr{P}_i(A) - \{\emptyset\} \subseteq D_s(\mathbf{A})$. □

3. Representation and duality for finite Hilbert algebras

It is known that for every finite Heyting algebra A or finite implicative semilattice (see [6]) there exists a finite poset X such that A is isomorphic to $\mathscr{P}_i(X)$.

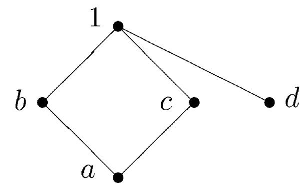

Fig. 1.

But in general this correspondence do not held for finite Hilbert algebras. In fact is not difficult to find a finite Hilbert algebra **A** such that neither $\mathbf{A} \cong \mathscr{P}_i(X)$ nor $\mathbf{A} \cong \mathscr{P}_i(X) - \{\emptyset\}$, for any finite poset X. For instance, Fig. 1 shows a finite Hilbert algebra $A = \{a, b, c, d, 1\}$ where the operation $\rightarrow$ is given by order, where

$$D_{\mathrm{ci}}(\mathbf{A}) = \{\{b, c, d, 1\}, \{c, d, 1\}, \{b, d, 1\}, \{a, b, c, 1\}\},$$

such that $\mathbf{A} \ncong \mathscr{P}_i(D_{\mathrm{ci}}(\mathbf{A})) - \{\emptyset\}$ and $\mathbf{A} \ncong \mathscr{P}_i(D_{\mathrm{ci}}(\mathbf{A}))$.

The next aim is to show that there exists a good representation for finite Hilbert algebras. The key of this representation is to give an adequate characterization of the image of the mapping $\varphi_{\mathbf{A}} : \mathbf{A} \rightarrow \mathscr{P}_i(D_{\mathrm{ci}}(\mathbf{A}))$. This is done by defining a non-empty subset $\mathscr{S}_{\mathbf{A}}$ of $\mathscr{P}(D_{\mathrm{ci}}(\mathbf{A}))$ and then defining a Hilbert algebra $\mathrm{H}(D_{\mathrm{ci}}(\mathbf{A})) \subset \mathscr{P}_i(D_{\mathrm{ci}}(\mathbf{A}))$ such that the image of $\varphi_{\mathbf{A}}$ is exactly the algebra $\mathrm{H}(D_{\mathrm{ci}}(\mathbf{A}))$.

We will start studying the relation between completely irreducible deductive systems and irreducible elements in a finite Hilbert algebra. By Lemma 13 we have that $(p]^c$ is a completely irreducible deductive system, for every irreducible element p of a Hilbert algebra **A**. We will see that if **A** is a finite Hilbert algebra then every completely irreducible deductive system of **A** is of the form $(p]^c$, for some irreducible element p.

For the rest of the paper we will note by $\mathbb{H}_{\mathrm{f}}$ the class of finite Hilbert algebras.

Lemma 16. *Let* $\mathbf{A} \in \mathbb{H}_{\mathrm{f}}$. *Let* $D \in D_{\mathrm{s}}(\mathbf{A})$. *Then* $D \in D_{\mathrm{ci}}(\mathbf{A})$ *if and only if there exists* $p_D \in A$ *such that* $D = (p_D]^c$.

Proof. Let $D \in D_{\mathrm{ci}}(\mathbf{A})$. Let us consider the set $K(D) = \{a \in A : (D, a) \in K\}$. It is clear that $K(D) \subseteq D^c$. Since $K(D)$ is a finite subset of A and D is an irreducible deductive system, we can deduce by Lemma 9 that there exists $p_D \notin D$ such that $a \leqslant p_D$ for every $a \in K(D)$. We will see that $p_D \in K(D)$. If $p_D \notin K(D)$, then there exists $P \in D_{\mathrm{ci}}(A)$ associated to p_D and $D \subsetneq P$. Since for every $a \in K(D)$, $a \leqslant p_D$, then $a \notin P$. It follows that D is not associated with a which is a contradiction by the fact that $a \in K(D)$. Thus $p_D \in K(D) \subseteq D^c$. Clearly $(p_D] \subseteq D^c$. Then $D \subseteq (p_D]^c$.

Let $b \notin D$. Since $p_D \notin D$ and $D \in D_{\mathrm{ci}}(\mathbf{A})$, then by item 2 of Theorem 5 there exists $c \notin D$ such that $b \leqslant c$ and $p_D \leqslant c$. So, $c \in K(D)$, and this implies $c \leqslant p_D$. It follows that $c = p_D$ and $b \leqslant p_D$. Therefore, $(p_D]^c \subseteq D$. This proves that $D = (p_D]^c$.

For the converse, note that if there exists $p_D \in A$ such that $D = (p_D]^c$, then D is maximal relative to p_D. Thus by Theorem 6 and Remark 14, $D \in D_{\mathrm{ci}}(\mathbf{A})$. $\square$

Clearly for every $D \in D_{\mathrm{ci}}(\mathbf{A})$, the element p_D such that $D = (p_D]^c$ is unique, and by Lemma 13 we have that p_D is irreducible.

From the previous proof we can deduce that if $a \in K(D)$, then $[a) \cap D^c \subseteq K(D)$.

In Example 3 we obtain a Hilbert algebra from a poset. Now we will see how to construct a Hilbert algebra from a poset endowed with a subset of power set.

Theorem 17. *Let $\langle X, \leqslant\rangle$ and $\emptyset \neq \mathscr{S} \subseteq \mathscr{P}(X)$. Then $\langle \mathrm{H}(X), \longmapsto, X\rangle$ is a Hilbert algebra, where*

$$\mathrm{H}(X) = \{U \in \mathscr{P}(X) : \textit{there exist } W \in \mathscr{S} \textit{ and } V \subseteq W, \textit{ such that } U = W \longmapsto V\}.$$

Proof. If $X=\emptyset$, then $\mathscr{S}=\{\emptyset\}$ and $\mathrm{H}(X)=\{\emptyset\}$. Thus clearly $\langle \mathrm{H}(X), \longmapsto, X\rangle$ is isomorphic to the trivial Hilbert algebra.

Now suppose that $X \neq \emptyset$. It is easy to see that $\mathrm{H}(X) \subseteq \mathscr{P}_i(X)$. We will prove that $\langle \mathrm{H}(X), \longmapsto, X\rangle$ is a subalgebra of the Hilbert algebra $\langle \mathscr{P}_i(X), \longmapsto, X\rangle$. Clearly $X \in \mathrm{H}(X)$, because $X = W \longmapsto W$ for any $W \in \mathscr{S}$.

Let $H, G \in \mathrm{H}(X)$. Then there exist $W_1, W_2 \in \mathscr{S}$ and $V_1 \subseteq W_1$ and $V_2 \subseteq W_2$ such that $H = W_1 \longmapsto V_1$ and $G = W_2 \longmapsto V_2$. We will prove that

$$H \longmapsto G = W_2 \longmapsto (W_2 \cap (H \longmapsto G)).$$

Let $x \in H \longmapsto G$. As $H \longmapsto G$ is increasing, $[x) \subseteq H \longmapsto G$. Therefore, $[x) \cap W_2 \subseteq W_2 \cap (H \longmapsto G)$ and consequently $x \in W_2 \longmapsto (W_2 \cap (H \longmapsto G))$.

Let $x \in W_2 \longmapsto (W_2 \cap (H \longmapsto G))$. Let $y \in H \cap [x)$. We will prove that $y \in G = W_2 \longmapsto V_2$. As $x \leqslant y$,

$$[y) \cap W_2 \subseteq [x) \cap W_2 \subseteq W_2 \cap (H \longmapsto G) \subseteq H \longmapsto G.$$

Since $y \in H$ and H is increasing,

$$[y) \cap W_2 \subseteq H \cap (H \longmapsto G) = H \cap G \subseteq G = W_2 \longmapsto V_2.$$

It follows that $[y) \cap W_2 \subseteq V_2$. Thus $x \in H \longmapsto G$. Then $H \longmapsto G = W_2 \longmapsto (W_2 \cap (H \longmapsto G)) \in \mathrm{H}(X)$. Finally, using the fact that $\mathbb{H}$ is a variety we can deduce that $\langle \mathrm{H}(X), \longmapsto, X\rangle$ is a Hilbert algebra. $\square$

Now we define the structures that will be the dual spaces of Hilbert algebras.

Definition 18. A triple $\langle X, \leqslant, \mathscr{S}\rangle$ is called an *Hilbert space* (**H**-*space* for short) if it satisfies the following properties:

HS1. $\langle X, \leqslant\rangle$ is a poset and $\emptyset \neq \mathscr{S} \subseteq \mathscr{P}(X)$.
HS2. For every $x \in X$, $\{x\} \in \mathscr{S}$.
HS3. For every $W \in \mathscr{S}$ and every $x, y \in W$, if $x \leqslant y$, then $x = y$.

By Theorem 17 we have that if $\langle X, \leqslant, \mathscr{S}\rangle$ is an **H**-space, then $\langle \mathrm{H}(X), \longmapsto, X\rangle$ is a Hilbert algebra called *the dual* of $\langle X, \leqslant, \mathscr{S}\rangle$.

If $X = \emptyset$, we will say that $\langle X, \leqslant \mathscr{S}\rangle$ is the trivial **H**-space. By Theorem 17 the dual of the trivial **H**-space is the trivial Hilbert algebra.

Now we will see how to obtain an **H**-space from a finite Hilbert algebra. Given $\mathbf{A} \in \mathbb{H}_{\mathrm{f}}$, we define

$$\mathscr{S}_{\mathbf{A}} = \{K^{-1}(a) : a \in A\},$$

where $K \subseteq D_{\mathrm{ci}}(\mathbf{A}) \times A$ is the relation defined in Definition 8.

Theorem 19. *Let* $\mathbf{A} \in \mathbb{H}_{\mathrm{f}}$. *Then the structure* $\langle D_{\mathrm{ci}}(\mathbf{A}), \subseteq, \mathscr{S}_{\mathbf{A}}\rangle$ *is an* **H**-*space*.

Proof. HS1. We note that $\{K^{-1}(1)\} = \{\emptyset\} \subseteq \mathscr{S}_{\mathbf{A}}$. Then $\mathscr{S}_{\mathbf{A}} \neq \emptyset$.

HS2. We note that from Lemma 16 we have that for every $D \in D_{\mathrm{ci}}(\mathbf{A})$, there exists one irreducible element $p_D \in A$ such that $D = (p_D]^c$. Thus $\{D\} = K^{-1}(p_D) \in \mathscr{S}_{\mathbf{A}}$.

HS3. If $a \in A$ and $P, Q \in K^{-1}(a)$ such that $P \subseteq Q$. Then $P = Q$, since $a \notin Q$ and P is maximal with respect to the deductive systems which do not contain a. $\square$

Note that if $\mathbf{A}$ is the trivial Hilbert algebra, then $\langle D_{\mathrm{ci}}(\mathbf{A}), \subseteq, \mathscr{S}_{\mathbf{A}}\rangle$ is the trivial **H**-space.

Theorem 20. *Let* $\mathbf{A} \in \mathbb{H}_{\mathrm{f}}$. *Then there exists an isomorphism between* $\mathbf{A}$ *and* $\mathrm{H}(D_{\mathrm{ci}}(\mathbf{A}))$.

Proof. If $\mathbf{A}$ is the trivial Hilbert algebra the results follows by the previous observations.

If $A \neq \{1\}$, we only need to prove that $\varphi_{\mathbf{A}}(\mathbf{A}) = \mathrm{H}(D_{\mathrm{ci}}(\mathbf{A}))$.

Let $F \in \mathrm{H}(D_{\mathrm{ci}}(\mathbf{A}))$. Then there exist $a \in A$ and $V \subseteq K^{-1}(a)$ such that

$$F = K^{-1}(a) \longmapsto V.$$

Since $\mathbf{A}$ is finite, $D_{\mathrm{ci}}(\mathbf{A})$ is finite too. Therefore $K^{-1}(a)$ is a finite set. From Lemma 16 there exists a subset $\{p_1, \ldots, p_n\} \subseteq A$ such that $V = \{(p_1]^c, \ldots, (p_n]^c\}$. We will prove that

$$F = \varphi_{\mathbf{A}}((p_1, \ldots, p_n; a)).$$

Let $P \in F = K^{-1}(a) \longmapsto V$. Suppose that $(p_1, \ldots, p_n; a) \notin P$. Then there exists $P_1 \in K^{-1}((p_1, \ldots, p_n; a))$ such that $P \subseteq P_1$. So, $a \notin \langle P_1 \cup \{p_1 \ldots, p_n\}\rangle$. It follows that there exists $P_a \in K^{-1}(a)$ such that $\langle P_1 \cup \{p_1, \ldots, p_n\}\rangle \subseteq P_a$. Then, $P \subseteq P_a$ and as $P_a \in K^{-1}(a)$, we get that $P_a \in [P) \cap K^{-1}(a)$. Then $P_a \in V$. Thus there exists p_i such that $P_a = (p_i]^c$, which is a contradiction, because $\{p_1, \ldots, p_n\} \subseteq P_a$. Therefore, $P \in \varphi_{\mathbf{A}}((p_1, \ldots, p_n; a))$.

Let $P \in \varphi_{\mathbf{A}}((p_1, \ldots, p_n; a))$. So $(p_1, \ldots, p_n; a) \in P$. We will prove that

$$[P) \cap K^{-1}(a) \subseteq V.$$

Let $Q \in [P) \cap K^{-1}(a)$. Then, $a \notin Q$ and $(p_1, \ldots, p_n; a) \in Q$. Thus there exists $1 \leqslant i \leqslant n$ such that $p_i \notin Q$. It follows that $Q \subseteq (p_i]^c$. But as $Q, (p_i]^c \in K^{-1}(a)$ and the elements of $K^{-1}(a)$ are incomparable, then $Q = (p_i]^c$. Thus, $Q \in V$. So, $P \in K^{-1}(a) \longmapsto V$.

Therefore, we have proved that $\mathrm{H}(D_{\mathrm{ci}}(\mathbf{A})) \subseteq \varphi_{\mathbf{A}}(\mathbf{A})$.

Now we will prove that $\varphi_{\mathbf{A}}(\mathbf{A}) \subseteq \mathrm{H}(D_{\mathrm{ci}}(\mathbf{A}))$. Let us prove that for each $a \in A$,

$$\varphi_{\mathbf{A}}(a) = K^{-1}(a) \longmapsto \emptyset.$$

Let $P \in \varphi_{\mathbf{A}}(a)$. If $Q \in D_{\mathrm{ci}}(\mathbf{A})$ and $P \subseteq Q$, $Q \notin K^{-1}(a)$, i.e., $[P) \cap K^{-1}(a) = \emptyset$. Consequently, $\varphi_{\mathbf{A}}(a) \subseteq K^{-1}(a) \longmapsto \emptyset$. Let $P \in K^{-1}(a) \longmapsto \emptyset$ and let us suppose that $a \notin P$. Then there exists $P_a \in K^{-1}(a)$ such that $P \subseteq P_a$. It follows that $P_a \in [P) \cap K^{-1}(a)$, which is a contradiction. Therefore, $a \in P$, i.e., $K^{-1}(a) \longmapsto \emptyset \subseteq \varphi_{\mathbf{A}}(a)$. □

In order to obtain a full duality between finite Hilbert algebras and finite **H**-spaces we need to define the notion of morphism between two **H**-spaces.

Definition 21. Let $\langle X_1, \leqslant_1, \mathscr{S}_1\rangle$ and $\langle X_2, \leqslant_2, \mathscr{S}_2\rangle$ be two **H**-spaces. Let us consider a relation $R \subseteq X_1 \times X_2$. We shall say that R is **H**-*functional* if it satisfies the following conditions:

HF1. If $(x, y) \in R$, there exists $z \in X_1$ satisfying $x \leqslant z$ and $R(z) = [y)$.
HF2. $(\leqslant_1 \circ R \circ \leqslant_2) \subseteq R$, where $\circ$ denotes the composition of relations.
HF3. For every $U \in \mathrm{H}(X_2)$,

$$h_R(U) = \{x \in X_1 : R(x) \subseteq U\} \in \mathrm{H}(X_2).$$

Now, we shall give some examples of **H**-functional relations that will be useful later.

Example 22. If $\langle X, \leqslant, \mathscr{S}\rangle$ is an **H**-space then we have that $\leqslant$ is an **H**-functional relation.

Example 23. Let $\langle X_1, \leqslant_1, \mathscr{S}_1\rangle$ and $\langle X_2, \leqslant_2, \mathscr{S}_2\rangle$ be two **H**-spaces. The empty relation is always an **H**-functional relation and for every $U \in \mathrm{H}(X_2)$, $h_R(U) = X_2$.

Example 24. Let $\langle X_1, \leqslant_1, \mathscr{S}_1\rangle$ and $\langle X_2, \leqslant_2, \mathscr{S}_2\rangle$ be two **H**-spaces. Let $f : X_1 \longrightarrow X_2$ be an increasing partial map such that:

1. $Dom(f) = \{x \in X_1 : \text{There exists } y \in X_2 \text{ such that } f(x) = y\}$ is a decreasing set,
2. If $x \in Dom(f)$ and $y \in X_2$ are such that $f(x) \leqslant_2 y$, then there exists $z \in Dom(f)$ such that $x \leqslant_1 z$ and $f(z) = y$.
3. For every $W_2 \in \mathscr{S}_2$, there exists $W_1 \in \mathscr{S}_1$ such that $f^{-1}(W_2) \subseteq W_1$.

We define

$$f^* = \{(x, y) \in X_1 \times X_2 : f(x) \leqslant y\}.$$

By condition 2, we have that f^* satisfies HF1 and that $\mathrm{Im}(f)$ is an increasing subset of X_2. Since f is an increasing map, using the facts that $Dom(f)$ is a decreasing set and $\mathrm{Im}(f)$ is an increasing set, we have that f^* satisfies HF2.

Let $W_2 \in \mathscr{S}_2$ and $V_2 \subseteq W_2$, and let $W_1 \in \mathscr{S}_1$ such that $f^{-1}(W_2) \subseteq W_1$. Let $V_1 = (W_1 \backslash f^{-1}(W_2)) \cup f^{-1}(V_2)$, then it is easy to see that

$$\begin{aligned}\{x \in X_1 : f^*(x) \subseteq W_2 \longmapsto V_2\} &= \{x \in X_1 : [f(x)) \subseteq W_2 \longmapsto V_2\}\\ &= \{x \in X_1 : [x) \subseteq W_1 \longmapsto V_1\}.\end{aligned}$$

Thus f^* is an **H**-functional relation.

Example 25. Let $\langle X_1, \leqslant_1, \mathscr{S}_1\rangle$ and $\langle X_2, \leqslant_2, \mathscr{S}_2\rangle$ be two **H**-spaces. Let $f : X_1 \longrightarrow X_2$ be an increasing bijective map such that:

1. For every $W_2 \in \mathscr{S}_2$ there exists $W_1 \in \mathscr{S}_1$ satisfying $f^{-1}(W_2) \subseteq W_1$, and
2. for every $W_1 \in \mathscr{S}_1$ there exists $W_2 \in \mathscr{S}_2$ such that $f(W_1) \subseteq W_2$.

Then f^* and $(f^{-1})^*$, defined as in the previous example, are **H**-functional relations satisfying $f^* \circ (f^{-1})^* = \leqslant_1$ and $(f^{-1})^* \circ f^* = \leqslant_2$.

We will see that the class of **H**-spaces with **H**-functional relations is a category (see [7]).

Given a category $\mathscr{L}$ and X, Y objects of this category, we will note by $\mathscr{L}(X, Y)$ the class of all morphism form X to Y.

If $R \subseteq X \times Y$ and $T \subseteq Y \times Z$ are **H**-functional relations between **H**-spaces then we define $T \bullet R \subseteq X \times Z$ by $T \bullet R = R \circ T$. Clearly $T \bullet R$ is an **H**-functional relation between X and Z.

$$\begin{tikzcd} X \arrow[r, dash, "R"] \arrow[dr, dash, "T \bullet R"'] & Y \arrow[d, dash, "T"] \\ & Z \end{tikzcd}$$

If $\langle X_1, \leqslant_1, \mathscr{S}_1\rangle$ and $\langle X_2, \leqslant_2, \mathscr{S}_2\rangle$ are **H**-spaces and $R \subseteq X_1 \times X_2$ is an **H**-functional relation then by HF2 of Definition 21 we have that $\leqslant_1 \circ R = R = R \circ \leqslant_2$. Thus the order relation in each **H**-space plays the role of the identity morphism. Then the class of **H**-spaces with **H**-functional relations is a category. We will note this category by $\mathscr{C}$, and by $\mathscr{C}_{\mathrm{f}}$ the full subcategory of $\mathscr{C}$ whose objects are finite **H**-spaces.

Let us recall that a *homomorphism* between two Hilbert algebras **A** and **B** is a map $h : \mathbf{A} \to \mathbf{B}$ such that $h(a \to b) = h(a) \to h(b)$, for every $a, b \in A$. We will note by $\mathscr{H}$ the category of Hilbert algebras and its homomorphisms, and by $\mathscr{H}_{\mathrm{f}}$ the full subcategory of $\mathscr{H}$ whose objects are finite Hilbert algebras.

Theorem 26. *Let $\langle X_1, \leqslant_1, \mathscr{S}_1\rangle$ and $\langle X_2, \leqslant_2, \mathscr{S}_2\rangle$ be two* **H**-*spaces and $R \in \mathscr{C}(X_1, X_2)$. Then the map $h_R \in \mathscr{H}(\mathrm{H}(X_2), \mathrm{H}(X_1))$, where $h_R : \mathrm{H}(X_2) \longrightarrow \mathrm{H}(X_1)$ is defined by*

$$h_R(U) = \{x \in X_1 : R(x) \subseteq U\},$$

for each $U \in \mathrm{H}(X_2)$.

Proof. By condition HF3 we have that h_R is well defined. Let $U, V \in \mathrm{H}(X_2)$. We need to prove that $h_R(U \longmapsto V) = h_R(U) \longmapsto h_R(V)$.

Let $x \in X_1$. Let us suppose that $x \in h_R(U \longmapsto V)$. Then $R(x) \subseteq U \longmapsto V$. We will prove that

$$[x) \cap h_R(U) \subseteq h_R(V).$$

Let $y \in [x) \cap h_R(U)$ and let $z \in X_2$ such that $(y, z) \in R$. By condition HF2 and taking into account that $\leqslant_1$ is reflexive, we have that $(x, z) \in R$. So, $z \in U \longmapsto V$, and since $z \in R(y) \subseteq U$, $z \in V$. Therefore $y \in h_R(V)$.

Let us suppose that $x \in h_R(U) \longmapsto h_R(V)$, i.e. $[x) \cap h_R(U) \subseteq h_R(V)$. Let $y \in R(x)$ and let $z \in X_2$ such that $y \leqslant_2 z$ and $z \in U$. Since $\leqslant_1$ is reflexive and by condition HF2 of Definition 21, we have that $(x, z) \in R$. By the condition HF1 of Definition 21, there exists $k \in X_1$ such that $x \leqslant_1 k$ and $R(k) = [z)$. Since $k \in [x)$ and $z \in U$, $k \in [x) \cap h_R(U) \subseteq h_R(V)$. It follows that $R(k) = [z) \subseteq V$, i.e., $z \in V$. Therefore, $x \in h_R(U \longmapsto V)$. □

Let **A** and **B** be two Hilbert algebras. Let $h : \mathbf{A} \to \mathbf{B}$ be a homomorphism. We define a relation $R_h \subseteq D_{\mathrm{si}}(\mathbf{B}) \times D_{\mathrm{si}}(\mathbf{A})$ as follows:

$$(Q, P) \in R_h \quad \text{if and only if} \quad h^{-1}(Q) \subseteq P.$$

In [2] the first author of this paper proves the next result.

Theorem 27. *Let* **A**, **B** *be two Hilbert algebras. Let* $h : A \to B$ *a function. Then* h *is a homomorphism if and only if the following conditions are held*:

1. *For every* $D \in D_{\mathrm{s}}(\mathbf{B})$, $h^{-1}(D) \in D_{\mathrm{s}}(\mathbf{A})$.
2. *For every* $(D, E) \in D_{\mathrm{si}}(\mathbf{B}) \times D_{\mathrm{si}}(\mathbf{A})$, *such that* $h^{-1}(E) \subseteq D$, *there exists* $Q \in D_{\mathrm{si}}(\mathbf{B})$ *such that* $h^{-1}(Q) = D$.

Theorem 28. *Let* **A**, **B** *be two finite Hilbert algebras. Let* $h \in \mathscr{H}_{\mathrm{f}}(\mathbf{A}, \mathbf{B})$. *Then* $R_h \in \mathscr{C}_{\mathrm{f}}(D_{\mathrm{ci}}(\mathbf{B}), D_{\mathrm{ci}}(\mathbf{A}))$.

Proof. Since **A**, **B** are finite Hilbert algebras, $D_{\mathrm{si}}(\mathbf{B}) \times D_{\mathrm{si}}(\mathbf{A}) = D_{\mathrm{ci}}(\mathbf{B}) \times D_{\mathrm{ci}}(\mathbf{A})$. From the previous theorem and from the results given in [2], it is easy to see that the relation R_h satisfies conditions HF1 and HF2.

Let $U \in \mathrm{H}(D_{\mathrm{ci}}(A))$. We need to prove that $h_{R_h}(U) \in \mathrm{H}(D_{\mathrm{ci}}(\mathbf{B}))$. By Theorem 20 there exists $a \in A$ such that $\varphi_{\mathbf{A}}(a) = U$. We will prove that $h_{R_h}(\varphi_{\mathbf{A}}(a)) = \varphi_{\mathbf{B}}(h(a))$. Let $P \in D_{\mathrm{ci}}(\mathbf{A})$. If $h(a) \in P$, then for any $Q \in D_{\mathrm{ci}}(\mathbf{B})$ such that $h^{-1}(P) \subseteq Q$, we have that $a \in Q$. Thus, $\varphi_{\mathbf{B}}(h(a)) \subseteq h_{R_h}(\varphi_{\mathbf{A}}(a))$. If there exists $P \in D_{\mathrm{ci}}(\mathbf{A})$ such that $R_h(P) \nsubseteq \varphi_{\mathbf{A}}(a)$, then there exists $Q \in D_{\mathrm{ci}}(\mathbf{B})$ such that $h^{-1}(P) \subseteq Q$ and $a \notin Q$. It follows that, $h(a) \notin P$. Therefore, $h_{R_h}(\varphi_{\mathbf{A}}(a)) \subseteq \varphi_{\mathbf{B}}(h(a))$. Then $h_{R_h}(U) \in \mathrm{H}(D_{\mathrm{ci}}(\mathbf{B}))$. So, R_h satisfies condition HF3. □

By the last proof we can deduce the next corollary.

Corollary 29. *Let* **A** *and* **B** *be two finite Hilbert algebras. Let* $h \in \mathscr{H}_{\mathrm{f}}(\mathbf{A}, \mathbf{B})$. *Let* R_h *be the* **H**-*functional relation associated to* h. *Let* h_{R_h} *be the homomorphism associated to* R_h. *Then*

$$h_{R_h} \circ \varphi_{\mathbf{A}} = \varphi_{\mathbf{B}} \circ h.$$

Our next task is to prove that $\mathscr{H}_{\mathrm{f}}$ is dually equivalent to $\mathscr{C}_{\mathrm{f}}$.

Lemma 30. *Let* $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *finite Hilbert algebras and* X, Y, Z *finite* **H**-*spaces then the following propositions hold*:

1. *If* $Id \in \mathscr{H}_{\mathrm{f}}(\mathbf{A}, \mathbf{A})$ *is the identity map*, *then* $R_{\mathrm{Id}} = \subseteq$.
2. *If* $h \in \mathscr{H}_{\mathrm{f}}(\mathbf{A}, \mathbf{B})$ *and* $g \in \mathscr{H}_{\mathrm{f}}(\mathbf{B}, \mathbf{C})$, *then* $R_{g \circ h} = R_h \bullet R_g$.
3. *If* $R \in \mathscr{C}_{\mathrm{f}}(X, X)$ *is the order relation in* X (*the identity morphism of* X *in* $\mathscr{C}_{\mathrm{f}}$), *then* h_R *is the identity map of* $\mathrm{H}(X)$.
4. *If* $R \in \mathscr{C}_{\mathrm{f}}(X, Y)$ *and* $S \in \mathscr{C}_{\mathrm{f}}(Y, Z)$, *then* $h_{S \bullet R} = h_R \circ h_S$.

Proof. 1. $R_{\mathrm{Id}} = \{(P, Q) \in D_{\mathrm{ci}}(\mathbf{B}) \times D_{\mathrm{ci}}(\mathbf{A}) : Id^{-1}(P) \subseteq Q\} = \{(P, Q) : P \subseteq Q\} = \subseteq$.

2. Suppose $(P, Q) \in R_{g \circ h}$, then $h^{-1}(g^{-1}(P)) \subseteq Q$. Since g is an homomorphism, $g^{-1}(P) \in D_{\mathrm{s}}(\mathbf{B})$ (see [2]). Then $g^{-1}(P) = \bigcap_{\substack{g^{-1}(P) \subseteq D \\ D \in D_{\mathrm{ci}}(\mathbf{B})}} D$. Thus

$$h^{-1}(g^{-1}(P)) = \bigcap_{\substack{g^{-1}(P) \subseteq D \\ D \in D_{\mathrm{ci}}(\mathbf{B})}} h^{-1}(D) \subseteq Q.$$

By the complete distributivity of deductive systems (see [5]) we have that there exists $D \in D_{\mathrm{ci}}(\mathbf{B})$ such that $g^{-1}(P) \subseteq D$ and $h^{-1}(D) \subseteq Q$. Thus $(P, Q) \in R_g \circ R_h = R_h \bullet R_g$.

The converse is easy and left to the reader.

3. If R is the order relation and $U \in \mathrm{H}(X)$ then

$$\begin{aligned} h_R(U) &= \{x \in X : R(x) \subseteq U\} \\ &= \{x \in X : [x) \subseteq U\} = U. \end{aligned}$$

4. Let $U \in \mathrm{H}(Z)$. Then

$$\begin{aligned} x \in (h_R \circ h_S)(U) &\Leftrightarrow R(x) \subseteq h_S(U) \Leftrightarrow S(R(x)) \subseteq U \\ &\Leftrightarrow x \in h_{S \bullet R}(U). \qquad \square \end{aligned}$$

By items 1 and 2 of Lemma 30 we can define the contravariant functor

$$\mathbf{D} : H_{\mathrm{f}} \longrightarrow C_{\mathrm{f}},$$

by $\mathbf{D}(\mathbf{A}) = \langle D_{\mathrm{ci}}(\mathbf{A}), \subseteq, \mathscr{S}_{\mathbf{A}} \rangle$ if $\mathbf{A} \in H_{\mathrm{f}}$, and $\mathbf{D}(h) = R_h$ if h is an homomorphism between finite Hilbert algebras.

By items 3 and 4 of Lemma 30 we can define the contravariant functor

$$\mathbf{H} : C_{\mathrm{f}} \longrightarrow H_{\mathrm{f}},$$

by $\mathbf{H}(\langle X, \leqslant, \mathscr{S} \rangle) = \langle \mathrm{H}(X), \longmapsto, X \rangle$ if $\langle X, \leqslant, \mathscr{S} \rangle \in C_{\mathrm{f}}$, and $\mathbf{H}(R) = h_R$ if R is an **H**-functional relations between finite **H**-spaces.

By Corollary 29 we have that φ is a natural equivalence between the identity functor of $\mathscr{H}_{\mathrm{f}}$ and $\mathbf{D} \circ \mathbf{H}$. Now we will prove that the identity functor of $\mathscr{C}_{\mathrm{f}}$ is naturally equivalent to $\mathbf{H} \circ \mathbf{D}$.

Lemma 31. *Let $\langle X, \leqslant, \mathscr{S}\rangle$ be a finite* **H**-*space. If $U, U_1, \ldots, U_n \in \mathrm{H}(X)$, then*

$$(U_1, \ldots, U_n; U) = \left(\bigcap_{i=1}^{n} U_i\right) \longmapsto U.$$

Proof. Let $U, U_1, U_2 \in \mathrm{H}(X)$. Then

$$\begin{aligned} U_1 \longmapsto (U_2 \longmapsto U) &= \{x \in X : [x) \cap U_1 \subseteq U_2 \longmapsto U\} \\ &= \{x \in X : [x) \cap (U_1 \cap U_2) \subseteq U\} \\ &= (U_1 \cap U_2) \longmapsto U. \end{aligned}$$

The rest follows by an easy induction argument over n. $\square$

Note that it is possible that $U_1 \cap U_2 \notin \mathrm{H}(X)$ but always $(U_1 \cap U_2) \longmapsto U = U_1 \longmapsto (U_2 \longmapsto U) \in \mathrm{H}(X)$.

Proposition 32. *Let $\langle X, \leqslant, \mathscr{S}\rangle$ be a finite* **H**-*space. Then there exists an increasing bijective map between $\langle X, \leqslant\rangle$ and $\langle D_{\mathrm{ci}}(\mathrm{H}(X)), \subseteq\rangle$.*

Proof. Let us consider the mapping $\varepsilon_X : X \longrightarrow D_{\mathrm{ci}}(\mathrm{H}(X))$ defined as follows:

$$\varepsilon_X(x) = \{U \in \mathrm{H}(X) : x \in U\}.$$

We prove that ε_X is well defined. Let $H, H \longmapsto G \in \varepsilon_X(x)$. So, $[x) \cap H \subseteq G$ and $x \in H$. Since H is increasing, $[x) \cap H = [x) \subseteq G$. Thus $x \in G$ and $G \in \varepsilon_X(x)$. Then $\varepsilon_X(x)$ is a deductive system of $\mathrm{H}(X)$.

Since $\langle X, \leqslant, \mathscr{S}\rangle$ is an **H**-space, $\{x\} \in \mathscr{S}$, for every $x \in X$. We will prove that

$$\varepsilon_X(x) = (\{x\} \longmapsto \emptyset]^c.$$

Let $x \in X$ and suppose that $U \in \mathrm{H}(X)$. Then,

$$\begin{aligned} U \in (\{x\} \longmapsto \emptyset]^c \quad &\Leftrightarrow \quad U \nsubseteq \{x\} \longmapsto \emptyset \quad \Leftrightarrow \quad U \cap \{x\} \neq \emptyset \\ &\Leftrightarrow \quad x \in U \quad \Leftrightarrow \quad U \in \varepsilon_X(x). \end{aligned}$$

Thus $\varepsilon_X(x) \in D_{\mathrm{ci}}(\mathrm{H}(X))$, for every $x \in X$.

Clearly by Remark 4 we have that

$$x \leqslant y \Leftrightarrow \varepsilon_X(x) \subseteq \varepsilon_X(x).$$

Then ε_X is an injective increasing map.

Let prove that ε_X is onto. Let $Q \in D_{\mathrm{ci}}(\mathrm{H}(X))$. Since $\mathrm{H}(X)$ is finite, by Lemma 16 we can deduce that there exists an irreducible element $U \in \mathrm{H}(X)$ such that $Q = (U]^c$. So, there exist $W \in \mathscr{S}$ and $V \subseteq W$ such that

$$\begin{aligned} U = W \longmapsto V = \{x \in X : [x) \cap W \subseteq V\} = \{x \in X : [x) \cap W \cap V^c = \emptyset\} \\ = (W \cap V^c) \longmapsto \emptyset = \bigcap_{x \in W \cap V^c} (\{x\} \longmapsto \emptyset). \end{aligned}$$

As $U \neq X$, $W \cap V^c$ is non-empty. Let us suppose that $W \cap V^c = \{x_1, \ldots, x_n\}$. Since

$$\bigcap_{i=1}^{n} (\{x_i\} \longmapsto \emptyset) = U,$$

by Lemma 31 we get $(\{x_1\} \longmapsto \emptyset, \ldots, \{x_n\} \longmapsto \emptyset; U) = 1$. Since U is irreducible, there exists i such that $(\{x_i\} \longmapsto \emptyset) \subseteq U$. It follows that $U = \{x_i\} \longmapsto \emptyset$. Therefore, $Q = \varepsilon_X(x)$, and ε_X is onto. $\square$

Theorem 33. *Let $\langle X, \leqslant, \mathscr{S}\rangle$ be a finite* **H**-*space. Then $(\varepsilon_X)^*$ is an isomorphism of* **H**-*spaces between $\langle X, \leqslant, \mathscr{S}\rangle$ and $\langle D_{\mathrm{ci}}(\mathrm{H}(X)), \subseteq, \mathscr{S}_{\mathrm{H}(X)}\rangle$.*

Proof. By Example 25 and the previous proposition we only have to prove that for every $W_2 \in \mathscr{S}_{\mathrm{H}(X)}$ there exists $W_1 \in \mathscr{S}$ satisfying $(\varepsilon_X)^{-1}(W_2) \subseteq W_1$, and for every $W_1 \in \mathscr{S}$ there exists $W_2 \in \mathscr{S}_{\mathrm{H}(X)}$ such that $\varepsilon_X(W_1) \subseteq W_2$.

Let $W_2 \in \mathscr{S}_{\mathrm{H}(X)}$. Then by Definition 8 there exist $W_1 \in \mathscr{S}$ and $V_1 \subseteq W_1$ such that $W_2 = K^{-1}(W_1 \longmapsto V_1)$.

We will prove that $(\varepsilon_X)^{-1}(W_2) \subseteq W_1$. Let $x \in (\varepsilon_X)^{-1}(W_2)$ such that $\varepsilon_X(x) = (\{x\} \longmapsto \emptyset]^c \in (K_{\mathrm{H}(X)})^{-1}(W_1 \longmapsto V_1)$. Then $W_1 \longmapsto V_1 \notin (\{x\} \longmapsto \emptyset]^c$ if and only if $W_1 \longmapsto V_1 \subseteq \{x\} \longmapsto \emptyset$. Thus $x \notin W_1 \longmapsto V_1$ if and only if $[x) \cap W_1 \nsubseteq V_1$. Let $y \in [x) \cap W_1 \cap (V_1)^c$ and suppose that $x < y$. Then by Remark 4 $(\{x\} \longmapsto \emptyset]^c \subsetneq (\{y\} \longmapsto \emptyset]^c$. But $(\{x\} \longmapsto \emptyset]^c \in (K_{\mathrm{H}(X)})^{-1}(W_1 \longmapsto V_1)$. So, $W_1 \longmapsto V_1 \in (\{y\} \longmapsto \emptyset]^c$ if and only if $W_1 \longmapsto V_1 \nsubseteq \{y\} \longmapsto \emptyset$. Then there exists $z \in X$, such that $[z) \cap W_1 \subseteq V_1$ and $z \leqslant y$. Thus, $[y) \cap W_1 \subseteq V_1$, which is contradiction with the fact that $y \in [y) \cap W_1$ and $y \in V_1^c$. Then $x = y \in W_1$. Therefore $(\varepsilon_X)^{-1}(W_2) \subseteq W_1 \in \mathscr{S}$.

Let $W_1 \in \mathscr{S}$ and $x \in W_1$. Then $\varepsilon_X(x) = (\{x\} \longmapsto \emptyset]^c \in (K_{\mathrm{H}(X)})^{-1}(W_1 \longmapsto \emptyset)$, because $W_1 \longmapsto \emptyset \subseteq \{x\} \longmapsto \emptyset$. If $y > x$, then $y \in W_1 \longmapsto \emptyset$ and $y \notin \{y\} \longmapsto \emptyset$. It follows that $W_1 \longmapsto \emptyset \in (\{y\} \longmapsto \emptyset]^c$. Therefore $\varepsilon_X(W_1) \subseteq (K_{\mathrm{H}(X)})^{-1}(W_1 \longmapsto \emptyset) \in \mathscr{S}_{\mathrm{H}(X)}$. $\square$

Proposition 34. *Let $\langle X, \leqslant_X, \mathscr{S}_X\rangle$ and $\langle Y, \leqslant_Y, \mathscr{S}_Y\rangle$ be two finite* **H**-*spaces and let R be an* **H**-*functional relation between X and Y. Then*

$$(x, y) \in R \quad \textit{if and only if} \quad (\varepsilon_X(x), \varepsilon_Y(y)) \in R_{h_R}.$$

Proof. Suppose that $(x, y) \in R$ and let $U \in \mathrm{H}(Y)$, such that $h_R(U) \in \varepsilon_X(x)$. Then $x \in h_R(U) = \{z \in X : R(z) \subseteq U\}$, concluding that $y \in R(x) \subseteq U$. Then $U \in \varepsilon_Y(y)$ so, $(h_R)^{-1}(\varepsilon_X(x)) \subseteq \varepsilon_Y(y)$. Therefore $(\varepsilon_X(x), \varepsilon_Y(y)) \in R_{h_R}$.

For the converse suppose that $(\varepsilon_X(x), \varepsilon_Y(y)) \in R_{h_R}$ and $(x, y) \notin R$. Then $R(x) \cap \{y\} = \emptyset$, which implies that $x \in h_R(\{y\} \longmapsto \emptyset)$. It follows that $\{y\} \longmapsto \emptyset \in (h_R)^{-1}(\varepsilon_X(x))$. Then $(h_R)^{-1}(\varepsilon_X(x)) \nsubseteq (\{y\} \longmapsto \emptyset]^c = \varepsilon_Y(y)$, which is a contradiction with the fact that $(\varepsilon_X(x), \varepsilon_Y(y)) \in R_{h_R}$. Thus $(x, y) \in R$. $\square$

Theorem 35. *Let $\langle X, \leqslant_X, \mathscr{S}_X\rangle$ and $\langle Y, \leqslant_Y, \mathscr{S}_Y\rangle$ be two finite* **H**-*spaces and let R be an* **H**-*functional relation between X and Y. Then*

$$(\varepsilon_Y)^* \bullet R = R_{h_R} \bullet (\varepsilon_X)^*.$$

Proof. It follows directly from Proposition 34 and the definition of the composition of H-functional relations. □

By Theorem 35 we have that ε^* is a natural equivalence between the identity functor in C_{f} and $\mathbf{D} \circ \mathbf{H}$. Then we can resume the results of this section in the following theorem.

Theorem 36. *The contravariant functors* **D** *and* **H** *defines a duality between the category of finite Hilbert algebras with homomorphisms and the category of finite* **H**-*spaces with* **H**-*functional relations.*

4. Ideal H-spaces

In this section we will study some properties of **H**-spaces, and we will define the notion of ideal **H**-space which will be useful later.

If $\langle X, \leqslant, \mathscr{S}\rangle$ is an **H**-space we have that for every $W \in \mathscr{S}$, $W \longmapsto \emptyset \in \mathrm{H}(X)$. However, the fact that there exists $W \in S$ such that $U = W \longmapsto \emptyset$, for some $U \in \mathrm{H}(X)$, is not necessarily always valid. In the next example we will see two isomorphic **H**-spaces which one of them has this special property and the other not.

Example 37. Let $\langle X, \leqslant\rangle$ be the poset given in Fig. 2.

Let us consider the subsets of $\mathscr{P}(X)$ defined by

$$\mathscr{S}_1 = \{\{a\}, \{b\}, \{c\}, \{d\}, \{a, b, c\}\}$$

and

$$\mathscr{S}_2 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{c, d\}, \{a, b, c\}\}.$$

It is easy to see that $\langle X, \leqslant, \mathscr{S}_1\rangle$ and $\langle X, \leqslant, \mathscr{S}_2\rangle$ are **H**-spaces.

It is clear that for every $W_2 \in \mathscr{S}_2$ there exists $W_1 \in \mathscr{S}_1$ such that $W_2 \subseteq W_1$, and conversely for every $W_1 \in \mathscr{S}_1$ there exists $W_2 \in \mathscr{S}_2$ such that $W_1 \subseteq W_2$. Then the function

$$f : X \longrightarrow X$$

defined by $f(x) = x$ satisfies the hypothesis of Example 25. Thus $f^* \subseteq X \times X$ is an isomorphism of **H**-spaces. By the duality given in the precious section we have that $\mathrm{H}\langle X, \leqslant, \mathscr{S}_1\rangle$

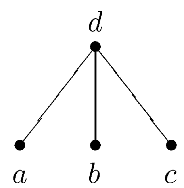

Fig. 2.

is isomorphic to $\mathrm{H}\langle X, \leqslant, \mathscr{S}_2\rangle$. Moreover, $\mathrm{H}\langle X, \leqslant, \mathscr{S}_1\rangle = \mathrm{H}\langle X, \leqslant, \mathscr{S}_2\rangle = \mathrm{H}(X)$. Now consider

$$\{a, b, c\} \longmapsto \{a\} = \{a, d\} \in \mathrm{H}(X).$$

We can check that for every $W \in \mathscr{S}_1$

$$\{a, d\} \neq W \longmapsto \emptyset.$$

But

$$\{a, d\} = \{b, c\} \longmapsto \emptyset,$$

with $\{b, c\} \in \mathscr{S}_2$. It is easy to see that for every $U \in \mathrm{H}(X)$ there exists $W \in \mathscr{S}_2$ such that $U = W \longmapsto \emptyset$.

The **H**-space $\langle X, \leqslant, \mathscr{S}_2\rangle$ of the previous example is a particular case of the following theorem.

Theorem 38. *Let $\langle X, \leqslant, \mathscr{S}\rangle$ be an* **H**-*space, which satisfies the following property*:

$$\textit{if } W \in \mathscr{S} \quad \textit{and} \quad V \subseteq W \quad \textit{implies that } V \in \mathscr{S}.$$

Then for every $U \in \mathrm{H}(X)$ there exists $Z \in \mathscr{S}$, such that $U = Z \longmapsto \emptyset$.

Proof. Let $U \in \mathrm{H}(X)$. Then there exist $W \in \mathscr{S}$ and $V \subseteq W$, such that

$$\begin{aligned} U = W \longmapsto V &= \{x \in X : [x) \cap W \subseteq V\} \\ &= \{x \in X : [x) \cap W \cap V^c = \emptyset\} \\ &= \{x \in X : [x) \cap (W \backslash V) = \emptyset\} \\ &= (W \backslash V) \longmapsto \emptyset. \end{aligned}$$

Clearly $(W\backslash V) \subseteq W$. Then if we set $Z = (W\backslash V)$, $Z \in \mathscr{S}$ and $U = Z \longmapsto \emptyset$. □

The above theorem motives the following definition.

Definition 39. Let $\langle X, \leqslant, \mathscr{S}\rangle$ be an **H**-space. We will say that $\langle X, \leqslant, \mathscr{S}\rangle$ is an *ideal* **H**-*space* (**IH**-space for short) if and only if it satisfies the following condition:

IH. If $W \in \mathscr{S}$, and $V \subseteq W$, then $V \in \mathscr{S}$.

In Example 37 we note that the **H**-space $\langle X, \leqslant, \mathscr{S}_1\rangle$ is isomorphic to the **IH**-space $\langle X, \leqslant, \mathscr{S}_2\rangle$. Following this idea we will prove that every **H**-space is isomorphic to some **IH**-space.

Let $\langle X, \leqslant, \mathscr{S}\rangle$ be an **H**-space. We define

$$I(\mathscr{S}) = \{V \in P(X) : \text{there exists } W \in \mathscr{S}, \text{ such that } V \subseteq W\}.$$

Note that $I(\mathscr{S})$ is the decreasing set generated by $\mathscr{S}$ in the poset $\langle \mathscr{P}(X), \subseteq\rangle$. It is easy to see that $\langle X, \leqslant, I(\mathscr{S})\rangle$ is an **IH**-space.

Theorem 40. *Let* $\langle X, \leqslant, \mathscr{S}\rangle$ *be an* **H**-*space. Then* $\langle X, \leqslant, \mathscr{S}\rangle$ *is isomorphic to* $\langle X, \leqslant, I(\mathscr{S})\rangle$.

Proof. Let

$$Id_X : X \longrightarrow X,$$

defined by $Id_X(x) = x$. If $W \in \mathscr{S}$, $f(W) = W \in \mathscr{S} \subseteq I(\mathscr{S})$. If $W_2 \in I(\mathscr{S})$, there exists $W_1 \in \mathscr{S}$ such that $(Id_X)^{-1}(W_2) = W_2 \subseteq W_1$. Then by Example 25 we have that $(Id_X)^* \subseteq X \times X$ is an isomorphism between $\langle X, \leqslant, \mathscr{S}\rangle$ and $\langle X, \leqslant, I(\mathscr{S})\rangle$. $\square$

Note that $(Id)^* = \leqslant$, but in this case it is an **H**-functional relation between two different **H**-spaces in the category $\mathscr{C}$ and not the identity morphism of $\langle X, \leqslant, \mathscr{S}\rangle$.

We will denote $\mathscr{I}\mathscr{H}$ the full subcategory of $\mathscr{C}$ whose spaces are ideal **H**-spaces. We define

$$\mathbf{I} : \mathscr{C} \longrightarrow \mathscr{I}\mathscr{H},$$

by

$$\mathbf{I}(\langle X, \leqslant, \mathscr{S}\rangle) = \langle X, \leqslant, I(\mathscr{S})\rangle$$

when $\langle X, \leqslant, \mathscr{S}\rangle$ is an **H**-space, and

$$\mathbf{I}(R) = R,$$

when R is an **H**-functional relation. It is easy to see that **I** is a functor. If $\mathbf{K} : \mathscr{I}\mathscr{H} \longrightarrow \mathscr{C}$ is the inclusion functor (see [7, p. 15]), by Theorem 40 ($\mathbf{I}$, $\mathbf{K}$, $(Id)^*$, 1) is an equivalence of categories $\mathscr{I}\mathscr{H}$ and $\mathscr{C}$ (see [7, Proposition 2, p. 92]), where 1_X is the identity morphism of X in $\mathscr{I}\mathscr{H}$. This proves that $\mathscr{I}\mathscr{H}$ is a reflective, and coreflective subcategory of $\mathscr{C}$.

We will note by $\mathscr{I}\mathscr{H}_{\mathrm{f}}$ the full subcategory of $\mathscr{I}\mathscr{H}$ whose spaces are finite.

Theorem 41. *Let* $\mathbf{A} \in \mathbb{H}_{\mathrm{f}}$. *Then* $\langle D_{\mathrm{ci}}(A), \subseteq, \mathscr{S}_{\mathbf{A}}\rangle$ *is an* **IH**-*space*.

Proof. Let $W \in \mathscr{S}_{\mathbf{A}}$ and $V \subseteq W$. Now consider $W \longmapsto (W \backslash V) \in \mathrm{H}(D_{\mathrm{ci}}(\mathbf{A}))$. By Theorem 20 there exists $a \in A$ such that $\varphi_{\mathbf{A}}(a) = W \longmapsto (W\backslash V)$. We will prove that $V = (K_{\mathbf{A}})^{-1}(a)$, concluding that $V \in \mathscr{S}_{\mathbf{A}}$.

First suppose that $V = \emptyset$, then $(W\backslash V) = W$, and $W \longmapsto (W\backslash V) = D_{\mathrm{ci}}(\mathbf{A})$. Thus $a = 1$, clearly $K^{-1}(1) = \emptyset = V$.

Now suppose that $V \neq \emptyset$. First note that if $Q \in V$ then $[Q) \cap W \nsubseteq W\backslash V$. Thus $Q \notin \varphi_{\mathbf{A}}(a)$, i.e., $a \notin Q$.

Now let $P \in K^{-1}(a)$, then $P \notin \varphi_{\mathbf{A}}(a) = W \longmapsto (W\backslash V)$. Thus there exists $Q \in V$ such that $P \subseteq Q$. But by the previous observation $a \notin Q$ and because Q is associated with a, we have that $P = Q$. Then $P \in V$.

Now let $Q \in V$. Since $a \notin Q$, there exists $P \in K^{-1}(a)$ such that $Q \subseteq P$. But we already prove that $K^{-1}(a) \subseteq V$, then $P \in V$. Then by HS3 we have that $Q = P$. Thus $Q \in K^{-1}(a)$. $\square$

If we define

$$\mathbf{H_I} : \mathscr{I}\mathscr{H}_{\mathrm{f}} \longrightarrow \mathscr{H}_{\mathrm{f}}$$

the restriction of $\mathbf{H} : \mathscr{C}_{\mathrm{f}} \longrightarrow \mathscr{H}_{\mathrm{f}}$ to the subcategory $\mathscr{I}\mathscr{H}_{\mathrm{f}}$, and we define $(\varepsilon_{\mathbf{I}})_X : X \longrightarrow D_{\mathrm{ci}}(\mathrm{H}(X))$ by $(\varepsilon_{\mathbf{I}})_X = (\varepsilon)_X$ when X is an **IH**-space, by Theorems 41 and 36 we conclude that

Theorem 42. *The contravariant functors* **D** *and* $\mathbf{H_I}$ *define a duality between the category of finite Hilbert algebras with homomorphisms and the category of finite* **IH**-*spaces with* **H**-*functional relations*.

5. Finite Hilbert algebras with lattice operations

Now we will study the representation of finite Hilbert algebras when they have supremum or infimum (relative to the natural order).

Definition 43. An algebra $A = \langle A, \rightarrow, \vee, 1\rangle$ of type $(2, 2, 0)$ is a *Hilbert algebra with supremum or $H^{\vee}$-algebra* if $\langle A, \rightarrow, 1\rangle$ is a Hilbert algebra, $\langle A, \vee, 1\rangle$ is a join-semilattice, and $a \rightarrow b = 1$ if and only if $a \vee b = b$. Hilbert algebras with infimum, or $H^{\wedge}$-algebras, are defined dually.

We will note $\mathscr{H}^{\vee}$ ($\mathscr{H}^{\wedge}$) the category of $H^{\vee}$-algebras ($H^{\wedge}$-algebra) with its homomorphisms and $(\mathscr{H}^{\vee})_{\mathrm{f}}$ ($(\mathscr{H}^{\wedge})_{\mathrm{f}}$) the full subcategory of $\mathscr{H}^{\vee}$ ($\mathscr{H}^{\wedge}$) whose algebras are finite.

First we will study the representation and the duality for finite $H^{\vee}$-algebras. So, we shall study the representation and the duality for a particular class of finite $H^{\wedge}$-algebras called *Brouwerian semilattices*.

5.1. Duality for finite $H^{\vee}$-algebras

Let $\langle X, \leqslant\rangle$ be a poset. We will denote by $\mathscr{I}(X)$ the set of subsets of X such their elements are incomparable, i.e., $W \in \mathscr{I}(X)$ if and only if the order $\leqslant$ of X restricted to W is the equality.

Definition 44. Let $\langle X, \leqslant\rangle$ a poset and $\mathscr{W} \subseteq \mathscr{P}(X)$. Let $S, T \in \mathscr{W}$. We say that S *covers* T, in symbols $S \preccurlyeq T$, if and only if for every $x \in S$, there exists $y \in T$ such that $x \leqslant y$.

We note that the relation $\preccurlyeq$ is a pre-order defined on $\mathscr{W} \subseteq \mathscr{P}(X)$. It is easy to see that if $\mathscr{W} \subseteq \mathscr{I}(X)$, $\preccurlyeq$ is an order relation in $\mathscr{W}$.

Let $\mathbf{A} \in \mathbb{H}$ and let us consider the poset $\langle D_{\mathrm{ci}}(\mathbf{A}), \subseteq\rangle$. Since $\mathscr{S}_{\mathbf{A}} \subseteq \mathscr{I}(D_{\mathrm{ci}}(\mathbf{A}))$, the relation $\preccurlyeq$ is an order in $\mathscr{S}_{\mathbf{A}}$. Let $U, V \in \mathscr{S}_{\mathbf{A}}$, we will note $U \curlyvee V$ ($U \curlywedge V$) to the supremum (infimum) of U and V in the order $\preccurlyeq$ whenever it exists. Note that $U \preccurlyeq V$ if and only if $(U] \subseteq (V]$ if and only if $(V]^c \subseteq (U]^c$ if and only if $V \longmapsto \emptyset \subseteq U \longmapsto \emptyset$.

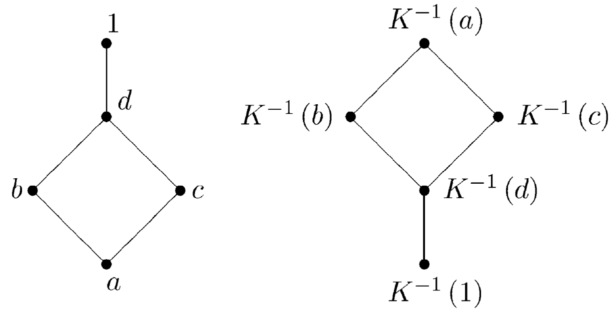

Fig. 3.

Now, we will see an example where the meet (join) in $S_{\mathbf{A}}$ does not coincide with intersection (union). Fig. 3 shows a Hilbert algebra **A** given by the order and the poset $\langle \mathscr{S}_{\mathbf{A}}, \preccurlyeq \rangle$:

By Theorem 15 we get $K^{-1}(b) = \{\{c, d, 1\}\}$ and $K^{-1}(c) = \{\{b, d, 1\}\}$, but

$$K^{-1}(b) \curlywedge K^{-1}(c) = K^{-1}(d) = \{\{1\}\} \neq K^{-1}(b) \cap K^{-1}(c) = \emptyset$$

and

$$\begin{aligned} K^{-1}(b) \curlyvee K^{-1}(c) = K^{-1}(a) = \{\{b, c, d, 1\}\} &\neq K^{-1}(b) \cup K^{-1}(c) \\ &= \{\{c, d, 1\}, \{b, d, 1\}\}. \end{aligned}$$

In this example we note that

$$K^{-1}(b) \curlywedge K^{-1}(c) = K^{-1}(b \vee c)$$

and

$$K^{-1}(b) \curlyvee K^{-1}(c) = K^{-1}(b \wedge c).$$

The next theorem shows the relationship between the order $\preccurlyeq$ in $\mathscr{S}_{\mathbf{A}}$ and the natural order of **A**.

Theorem 45. *Let* $\mathbf{A} \in \mathbb{H}$ *and* $a, b \in A$. *Then* $a \leqslant b$ *if and only if* $K^{-1}(b) \preccurlyeq K^{-1}(a)$.

Proof. Suppose first that $a \leqslant b$. Let $D \in K^{-1}(b)$, thus $a \notin D$. By Corollary 7 there exists $E \in K^{-1}(a)$ such that $D \subseteq E$. Then $K^{-1}(b) \preccurlyeq K^{-1}(a)$.

Suppose now that $K^{-1}(b) \preccurlyeq K^{-1}(a)$. If $a \nleqslant b$ there exists $D \in K^{-1}(b)$ such that $a \in D$. Since, $K^{-1}(b) \preccurlyeq K^{-1}(a)$, there exists $E \in K^{-1}(a)$ such that $D \subseteq E$. Then $a \in E \in K^{-1}(a)$, which is a contradiction. Thus, $a \leqslant b$. $\square$

By the previous theorem, we can conclude that there exists a supremum (infimum) relative to the natural order between two elements $a, b \in A$ if and only if there exists infimum (supremum) between $K^{-1}(a)$ and $K^{-1}(b)$ in $\langle \mathcal{S}_{\mathbf{A}}, \preccurlyeq \rangle$. Now we will prove the converse for ideal $\mathbf{H}$-spaces:

Theorem 46. *Let $\langle X, \leqslant, \mathcal{S} \rangle$ be an* **IH**-*space such that $\langle \mathcal{S}, \preccurlyeq \rangle$ is a meet-semilattice. Then $\langle \mathrm{H}(X), \longmapsto, \veebar, X \rangle$ is a $H^{\vee}$-algebra, where*

$$U \veebar V = (W_1 \curlywedge W_2) \longmapsto \emptyset,$$

for every $U, V \in \mathrm{H}(X)$, whenever $U = W_1 \longmapsto \emptyset$ and $V = W_2 \longmapsto \emptyset$.

Proof. Let $x \in U = W_1 \longmapsto \emptyset$, i.e., $[x) \cap W_1 = \emptyset$. Suppose that $x \notin (W_1 \curlywedge W_2) \longmapsto \emptyset$. Then $[x) \cap (W_1 \curlywedge W_2) \neq \emptyset$. Let $y \geqslant x$ such that $y \in W_1 \curlywedge W_2$. Since $W_1 \curlywedge W_2 \preccurlyeq W_1$ there exists $z \in W_1$ such that $y \leqslant z$. Then $z \in [x) \cap W_1$ which is a contradiction. Thus, $U \subseteq (W_1 \curlywedge W_2) \longmapsto \emptyset$. By a similar argument we have that $V \subseteq (W_1 \curlywedge W_2) \longmapsto \emptyset$.

Let $W \in \mathcal{S}$ such that $U, V \subseteq W \longmapsto \emptyset$. We will prove that $(W_1 \curlywedge W_2) \longmapsto \emptyset \subseteq W \longmapsto \emptyset$. If $x \in W$, then $x \notin W \longmapsto \emptyset$, and we have that $x \notin W_1 \to \emptyset$, i.e., $[x) \cap W_1 \neq \emptyset$. Thus there exists $y \in W_1$ such that $x \leqslant y$, i.e., $W \preccurlyeq W_1$. By the same argument $W \preccurlyeq W_2$. Thus $W \preccurlyeq W_1 \curlywedge W_2$. Then we conclude that $(W_1 \curlywedge W_2) \longmapsto \emptyset \subseteq W \longmapsto \emptyset$. $\square$

Let $\mathbf{A}$ be an $H^{\vee}$-algebra and let $D \in D_{\mathrm{s}}(\mathbf{A})$. We say that D is *prime* if and only if $D \neq A$ and for every $a, b \in A$ such that $a \vee b \in D$, $a \in D$ or $b \in D$. The next result is known but for the sake of completeness we give a proof.

Lemma 47. *Let $\mathbf{A}$ be an $H^{\vee}$-algebra and let $D \in D_{\mathrm{s}}(\mathbf{A})$. Then $D \in D_{\mathrm{si}}(\mathbf{A})$ if and only if D is prime.*

Proof. Let $D \in D_{\mathrm{si}}(\mathbf{A})$. Suppose that $a, b \in A$ and $a \vee b \in D$. If $a, b \notin D$ there exists $c \notin D$ such that $a, b \leqslant c$. But as $a \vee b \leqslant c \notin D$, which is a contradiction. Thus, D is prime. Let us suppose that D is prime. Let $a, b \notin D$. So, $a \vee b \notin D$ and as $a, b \leqslant a \vee b$, $D \in D_{\mathrm{si}}(\mathbf{A})$. $\square$

Theorem 48. *Let $\mathbf{A}, \mathbf{B}$ be two $H^{\vee}$-algebras and let $h : A \to B$ be a homomorphism of Hilbert algebras. Then h preserves the operation $\vee$ if and only if for every $Q \in D_{\mathrm{si}}(\mathbf{B})$, $h^{-1}(Q) \in D_{\mathrm{si}}(\mathbf{A})$ or $h^{-1}(Q) = A$.*

Proof. Suppose that h is a homomorphism of Hilbert algebras that preserves $\vee$ i.e., for every $a, b \in A$, $h(a \vee b) = h(a) \vee h(b)$. Let $Q \in D_{\mathrm{si}}(\mathbf{B})$ such that $h^{-1}(Q) \neq A$, and let $a \vee b \in h^{-1}(Q)$. Then $h(a) \vee h(b) \in Q$ and by the previous lemma Q is prime, then $a \in h^{-1}(Q)$ or $b \in h^{-1}(Q)$. So, $h^{-1}(Q)$ is prime. Again by the previous lemma we get $h^{-1}(Q) \in D_{\mathrm{si}}(\mathbf{A})$.

Let us assume that for every $Q \in D_{\mathrm{si}}(\mathbf{B})$, $h^{-1}(Q) \in D_{\mathrm{si}}(\mathbf{A}) \cup \{A\}$. Let us suppose that there exist $a, b \in A$ such that $h(a \vee b) \nleqslant h(a) \vee h(b)$. By Lemma 7 there exits $Q \in D_{\mathrm{si}}(\mathbf{B})$ such that $h(a \vee b) \in Q$ and $h(a) \vee h(b) \notin Q$. Then $a \vee b \in h^{-1}(Q)$ and $h^{-1}(Q) \neq A$. Thus $h(a) \in Q$ or $h(b) \in Q$, but this is a contradiction since $h(a) \vee h(b) \notin Q$. Then h preserves the operation $\vee$. $\square$

From the previous theorem we can conclude that if $\mathbf{A}$, $\mathbf{B}$ are two finite $H^{\vee}$-algebras and $h : A \longrightarrow B$ is a homomorphism of Hilbert algebras, then h preserves $\vee$ if and only if for every $Q \in D_{\mathrm{si}}(\mathbf{B})$, $R_h(Q) = [h^{-1}(Q))$ or $R_h(Q) = \emptyset$. So, in this case we can replace the relation R_h by a partial function $g_h : D_{\mathrm{si}}(\mathbf{B}) \longrightarrow D_{\mathrm{si}}(\mathbf{A})$ where the $Dom(g_h) = \{Q \in D_{\mathrm{si}}(\mathbf{B}) : h^{-1}(Q) \neq A\}$ and defined by: $g_h(Q) = h^{-1}(Q)$, for each $Q \in Dom(g_h)$, and it is clear that $R_h = (g_h)^*$ (see Example 24).

Corollary 49. *Let $\langle X, \leqslant_X, \mathscr{S}_X\rangle$ and $\langle Y, \leqslant_Y, \mathscr{S}_Y\rangle$ be two finite* **IH**-*spaces such that $\langle \mathscr{S}_X, \preccurlyeq_X\rangle$ and $\langle \mathscr{S}_Y, \leqslant_Y\rangle$ are meet-semilattices. Let $R \subseteq X \times Y$ an* **H**-*functional relation. Then $h_R : \mathrm{H}(Y) \longrightarrow \mathrm{H}(X)$ preserves the operation $\veebar$ if and only if for every $x \in X$ there exists $y \in Y$ such that $R(x) = [y)$ or $R(x) = \emptyset$.*

We define $\mathscr{I}\mathscr{H}^{\curlywedge}$ the subcategory of $\mathscr{I}\mathscr{H}_{\mathrm{f}}$ whose spaces $\langle X, \leqslant, \mathscr{S}\rangle$ are such $\langle \mathscr{S}, \preccurlyeq\rangle$ is a meet-semilattice and whose **H**-functional relations $R \subseteq X \times Y$ satisfy the condition that, for every $x \in X$ there exists $y \in Y$ such that $R(x) = [y)$ or $R(x) = \emptyset$.

Let

$$\mathbf{D}^{\vee} : (\mathscr{H}^{\vee})_{\mathrm{f}} \longrightarrow \mathscr{I}\mathscr{H}^{\curlywedge}$$

be the functor defined by

$$\mathbf{D}^{\vee}(\mathbf{A}) = \langle D_{\mathrm{ci}}(\mathbf{A}), \subseteq, \mathscr{S}_{\mathscr{A}}\rangle,$$
$$\mathbf{D}^{\vee}(h) = R_h,$$

where $\mathbf{A}$ is an $H^{\vee}$-algebra, and h is a homomorphism between $H^{\vee}-$algebras.

Let

$$\mathbf{H}^{\vee} : \mathscr{I}\mathscr{H}^{\curlywedge} \longrightarrow (\mathscr{H}^{\vee})_{\mathrm{f}}$$

be the functor defined by

$$\mathbf{H}^{\vee}(X) = \langle \mathrm{H}(X), \longmapsto, \veebar, X\rangle,$$
$$\mathbf{H}^{\vee}(R) = h_R,$$

where $\langle X, \leqslant, \mathscr{S}\rangle$ is an object of $\mathscr{I}\mathscr{H}^{\curlywedge}$ and R is a morphism of $\mathscr{I}\mathscr{H}^{\curlywedge}$.

Theorem 50. *The contravariant functors $\mathbf{D}^{\vee}$ and $\mathbf{H}^{\vee}$ define a duality between the category $(\mathscr{H}^{\vee})_{\mathrm{f}}$ and the category $\mathscr{I}\mathscr{H}^{\curlywedge}$.*

Proof. We only have to prove that for every $\mathbf{A}$ $H^{\vee}$-algebra and every $a, b \in A$

$$\varphi_{\mathbf{A}}(a \vee b) = \varphi_{\mathbf{A}}(a) \veebar \varphi_{\mathbf{A}}(b).$$

By the observation after Theorem 45 we have that $K^{-1}(a \vee b) = K^{-1}(a) \curlywedge K^{-1}(b)$. Then

$$\begin{aligned}\varphi_{\mathbf{A}}(a \vee b) &= K^{-1}(a \vee b) \longmapsto \emptyset\\ &= (K^{-1}(a) \curlywedge K^{-1}(b)) \longmapsto \emptyset\\ &= (K^{-1}(a) \longmapsto \emptyset) \veebar (K^{-1}(b) \longmapsto \emptyset)\\ &= \varphi_{\mathbf{A}}(a) \veebar \varphi_{\mathbf{A}}(b). \qquad \square\end{aligned}$$

5.2. Duality for finite Brouwerian semilattices

A particular class of $H^{\wedge}$-algebras are the called *Brouwerian semilattices* (see [6]), also known as *Hertz algebras* (see [10]) or *implicative semilattices* (see [8]). In Köhler [6], using the concept of meet-irreducible element, gives a duality between finite posets and finite Brouwerian semilattices. Here we will study the relationship between his duality and the duality given in Section 3.

Definition 51. A *Brouwerian semilattice* is an algebra $A = \langle A, \wedge, \rightarrow, 1\rangle$ of type (2, 2, 0) such that the following conditions are satisfied for every $a, b, c \in A$:

B1. $a \rightarrow a = 1$.
B2. $(a \rightarrow b) \wedge b = b$.
B3. $a \wedge (a \rightarrow b) = a \wedge b$.
B4. $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$.

The class of Brouwerian semilattices, form a variety, we will note it by $\mathbb{BS}$. It is clear that if $\mathbf{A} \in \mathbb{BS}$, then the reduct $\langle A, \rightarrow, 1\rangle$ is a Hilbert algebra and the reduct $\langle A, \wedge, 1\rangle$ is a meet-semilattice. A subset $F \subseteq A$ is called a *filter* of $\mathbf{A}$ if F is an increasing set, and for every $a, b \in F$, $a \wedge b \in F$. We will denote by $F(\mathbf{A})$ the set of all filters of $\mathbf{A}$. It is known that $F(\mathbf{A}) = D_s(\mathbf{A})$ (see [8]).

We note that there exists Hilbert algebras with a structure of meet-semilattice relative to the natural ordering such that they are not Brouwerian semilattices. For example, the Boolean lattice with two atoms $\{0, a, b, 1\}$ with the implication $\rightarrow$ defined by the order $\leqslant$ (Definition 11) is a Hilbert algebra where the infimum exists for any pair of elements but does not satisfy the identity B4 of Definition 51, because

$$(a \rightarrow a) \wedge (a \rightarrow b) = 1 \wedge b = b \neq 0 = a \rightarrow (a \wedge b) = a \rightarrow 0.$$

Let $\langle X, \leqslant\rangle$ be a poset. It is clear that the algebra $\langle \mathscr{P}_i(X), \cap, \longmapsto, X\rangle \in \mathbb{BS}$. In general, if $\langle X, \leqslant, \mathscr{S}\rangle$ is a finite $\mathbf{H}$-space, the Hilbert algebra $\mathrm{H}(X)$ is not closed under $\cap$.

Theorem 52. *Let $\langle X, \leqslant, \mathscr{S}\rangle$ be a finite $\mathbf{H}$-space. Suppose that $W \cup K \in \mathscr{S}$, for every $W, K \in \mathscr{S}$. Then, $\mathrm{H}(X) = \mathscr{P}_i(X)$. Thus, $\langle \mathrm{H}(X), \cap, \longmapsto, X\rangle$ is a Brouwerian semilattice.*

Proof. We prove that $X \in \mathscr{S}$. Since X is finite, $X = \{x_1, \ldots, x_n\} = \{x_1\} \cup \cdots \cup \{x_n\}$, and as $\langle X, \leqslant, \mathscr{S}\rangle$ is an $\mathbf{H}$-space, $\{x\} \in \mathscr{S}$, for every $x \in X$. Thus, by assumption, $X = \{x_1\} \cup \cdots \cup \{x_n\} \in \mathscr{S}$. Let $U \in \mathscr{P}_i(X)$. As $U = X \longmapsto U$ and $X \in \mathscr{S}$, $U \in \mathrm{H}(X)$. Now, since $\mathscr{P}_i(X)$ is closed under $\cap$, $\langle \mathrm{H}(X), \cap, \longmapsto, X\rangle$ is a Brouwerian semilattice. $\square$

Lemma 53. *Let $\mathbf{A}$ be a finite Brouwerian semilattice. Then $\mathscr{S}_{\mathbf{A}} = \mathscr{I}(D_{\mathrm{ci}}(\mathbf{A}))$.*

Proof. Clearly $\mathscr{S}_{\mathbf{A}} \subseteq \mathscr{I}(D_{\mathrm{ci}}(\mathbf{A}))$. Let $H \in \mathscr{I}(D_{\mathrm{ci}}(\mathbf{A}))$. By Lemma 16, there exists $\{p_1, \ldots, p_n\} \subseteq A$, such that $H = \{(p_1]^c, \ldots, (p_n]^c\}$. If $i, j \leqslant n$ and $i \neq j$, then $p_i \in (p_j]^c$, because otherwise $p_i \leqslant p_j$, i.e., $(p_j]^c \subseteq (p_i]^c$, which is a contradiction, since $H \in \mathscr{I}(D_{\mathrm{ci}}(\mathbf{A}))$.

Let $a = \bigwedge_{1 \leqslant i \leqslant n} p_i$. We will prove that $H = K^{-1}(a)$. Let $(p_i]^c \in H$. Obviously $a \notin (p_i]^c$. Let $p \notin (p_i]^c$. We prove that $p \to a \in (p_i]^c$. As $(p_i]^c$ is associated to p_i, $p \to p_i \in (p_i]^c$. On the other hand, every $j \neq i$, $p_j \leqslant p \to p_j \in (p_i]^c$. Thus,

$$p \to a = p \to \bigwedge_{1 \leqslant j \leqslant n} p_j = \bigwedge_{1 \leqslant j \leqslant n} (p \to p_j) \in (p_i]^c,$$

and by Theorem 6, $(p_i]^c \in K^{-1}(a)$. Then $H \subseteq K^{-1}(a)$.

Let $D \in K^{-1}(a)$. Since $a = \bigwedge_{1 \leqslant i \leqslant n} p_i \notin D$, there exists at least one i such that $p_i \notin D$. Then, $D \subseteq (p_i]^c$. As $a \notin D$ and $a \notin (p_i]^c$, by maximality of D, we get $D = (p_i]^c$. Then $K^{-1}(a) \subseteq H$. Therefore $H = K^{-1}(a)$. □

Theorem 54. *Let* $\mathbf{A} \in \mathbb{H}_{\mathrm{f}}$. *Then* $\mathbf{A}$ *is the implicative reduct of a Brouwerian semilattice if and only if* $\mathrm{H}(D_{\mathrm{ci}}(\mathbf{A})) = \mathscr{P}_i(D_{\mathrm{ci}}(\mathbf{A}))$.

Proof. If $\mathrm{H}(D_{\mathrm{ci}}(\mathbf{A})) = \mathscr{P}_i(D_{\mathrm{ci}}(\mathbf{A}))$ and taking into account that

$$\langle \mathbf{A}, \to, 1 \rangle \cong \langle \mathrm{H}(D_{\mathrm{ci}}(\mathbf{A})), \longmapsto, D_{\mathrm{ci}}(\mathbf{A}) \rangle,$$

then clearly $\mathbf{A}$ is the implicative reduct of a Brouwerian semilattice.

Suppose that $\mathbf{A}$ is the implicative reduct of a finite Brouwerian semilattice $\mathbf{B}$. Let $H \in \mathscr{P}_i(D_{\mathrm{ci}}(\mathbf{A}))$. Since H is finite, then $D_{\mathrm{ci}}(\mathbf{A}) - H$ is finite too, and consequently the set of all maximal elements of $D_{\mathrm{ci}}(\mathbf{A}) - H$ is a finite subset. Let us call this set M. From the previous lemma, $M \in \mathscr{S}_{\mathbf{A}}$. Let us prove that $H = M \longmapsto \emptyset$. Indeed

$$\begin{aligned} M \longmapsto \emptyset &= \{D \in D_{\mathrm{ci}}(\mathbf{A}) : [D) \cap M = \emptyset\} \\ &= \{D \in D_{\mathrm{ci}}(\mathbf{A}) : [D) \subseteq D_{\mathrm{ci}}(\mathbf{A}) - M\} \\ &= \{D \in D_{\mathrm{ci}}(\mathbf{A}) : [D) \subseteq H\} \\ &= \{D \in D_{\mathrm{ci}}(\mathbf{A}) : D \in H\} = H. \end{aligned}$$

Therefore, $H \in \mathrm{H}(D_{\mathrm{ci}}(\mathbf{A}))$. □

Let us recall that an element p of a Brouwerian semilattice $\mathbf{A}$ is called *meet-irreducible* if $p \neq 1$, and if $p = q \wedge r$ implies that $p = q$ or $p = r$. Taking into account Lemma 2.1 of [6] we can deduce that the notion of irreducible element (see Definition 12) and meet-irreducible element are equivalent in Brouwerian semilattices. On the other hand, by Lemma 16 it is easy to see that if $\mathbf{A}$ is a finite Brouwerian semilattice then there exists a dual order-isomorphism between the set $M(\mathbf{A})$ of all irreducible elements of $\mathbf{A}$ and the set $D_{\mathrm{ci}}(\mathbf{A})$. If we denote by $\mathscr{P}_{\mathrm{d}}(M(\mathbf{A}))$ the set of all decreasing subsets of $M(\mathbf{A})$, then the dual order-isomorphism can be written as

$$\mathscr{P}_{\mathrm{d}}(M(\mathbf{A})) \cong \mathscr{P}_i(D_{\mathrm{ci}}(\mathbf{A})).$$

Now, from Theorem 54 we deduce that

$$\mathrm{H}(D_{\mathrm{ci}}(\mathbf{A})) \cong \mathscr{P}_{\mathrm{d}}(M(\mathbf{A})).$$

Moreover, if **A** is a finite Brouwerian semilattice, then by Lemma 53 we get $\mathscr{S}_{\mathbf{A}} = \mathscr{I}(D_{\mathrm{ci}}(\mathbf{A}))$, i.e., the set $\mathscr{S}_{\mathbf{A}}$ can be obtained by $D_{\mathrm{ci}}(\mathbf{A})$ and the inclusion order, therefore it does not give us new information and it can be omitted.

6. Negative results on natural dualities

Natural duality is a special type of duality defined over quasi-varieties $\mathbb{V}$ which are generated by a finite algebra **A** in the following way:

$$\mathbb{V} = \mathbb{ISP}(\mathbf{A}), \tag{1}$$

where $\mathbb{I}$, $\mathbb{S}$ and $\mathbb{P}$ denotes the class isomorphic copies, subalgebras and direct products, respectively. For quasi-varieties satisfying condition (1) we can obtain representations by mean of topological relational structures (see [4] for details), and in some special cases those representations are dualities.

In this section we will prove that for every finite Hilbert algebra **A**, $\mathbb{ISP}(\mathbf{A}) \subsetneq \mathbb{H}$.

If **A** is a Hilbert algebra, we denote $Con(\mathbf{A})$ the lattice of congruences of **A**. We say that a Hilbert algebra $\mathbf{A} = \langle A, \rightarrow, 1\rangle$ is non-trivial if $A \neq \{1\}$.

First we will recall the following result.

Theorem 55. *Let* $\mathbf{A} \in \mathbb{H}$. *The map* $\theta : D_{\mathrm{s}}(\mathbf{A}) \longmapsto Con(\mathbf{A})$ *defined by*

$$\theta(F) = \{(a, b) \in A^2 : a \rightarrow b, b \rightarrow a \in F\}$$

for every $F \in D_{\mathrm{s}}(A)$ *is a lattice isomorphism.*

Theorem 56. *Let* **A** *be a non-trivial Hilbert algebra. Then the following conditions are equivalent*:

1. **A** *is subdirectly irreducible*.
2. *There exists* $p \in A - \{1\}$, *such that* $a \leqslant p$, *for every* $a \in A - \{1\}$.
3. $\{1\} \in D_{\mathrm{ci}}(\mathbf{A})$.

Proof. $1 \Rightarrow 2$. As **A** is subdirectly irreducible there exists a least non-trivial congruence. By the previous theorem we have that there exists a least non-trivial deductive system M. Suppose that there exist $p, q \in M - \{1\}$. Since $[p), [q) \in D_{\mathrm{s}}(\mathbf{A})$ and $[p) \neq \{1\} \neq [q)$, $M \subseteq [p), [q)$. Thus $p \leqslant q$ and $q \leqslant p$. Then $M = \{1, p\}$ and $p \neq 1$. Let $a \in A - \{1\}$. Then $[a) \in D_{\mathrm{s}}(\mathbf{A}) - \{\{1\}\}$. Thus $p \in M \subseteq [a)$, i.e., $a \leqslant p$.

$2 \Rightarrow 3$. Clearly $\{1\}$ is associated with p, then $\{1\} \in D_{\mathrm{ci}}(\mathbf{A})$.

$3 \Rightarrow 1$. We will denote ω the trivial congruence of **A**. By the hypothesis we have that $\{1\} \neq \bigcap_{D \in D_{\mathrm{s}}(\mathbf{A}) - \{\{1\}\}} D$. Then by the previous theorem we get that $\omega \neq \bigcap_{\theta \in Cong(\mathbf{A}) - \{\omega\}} \theta$. Thus we have that $\bigcap_{\theta \in Cong(\mathbf{A}) - \{\omega\}} \theta$ is the least non-trivial congruence of **A**, and consequently **A** is subdirectly irreducible. $\square$

Definition 57. Let **A** a Hilbert algebra and p an element such that $p \notin A$. Then $\mathbf{A}^+ = \langle A^+, \to^+, 1\rangle$ is called *the amplied Hilbert algebra of* **A**, where $A^+ = A \cup \{p\}$ and

$$a \to^+ b = \begin{cases} a \to b & \text{if } a, b \in A, \\ 1 & \text{if } a \in A - \{1\} \text{ and } b = p, \\ b & \text{if } b \in A - \{1\} \text{ and } a = p, \\ p & \text{if } a = 1 \text{ and } b = p, \\ 1 & \text{if } a = p \text{ and } b = 1. \end{cases}$$

In Diego [5] proves that for every Hilbert algebra **A**, $\mathbf{A}^+$ is a Hilbert algebra. We note that by Theorem 56, $\mathbf{A}^+$ is always subdirectly irreducible.

Theorem 58. *Let* **A** *be a finite Hilbert algebra, then* $\mathbf{A}^+ \notin \mathbb{ISP}(\mathbf{A})$.

Proof. Suppose that $\mathbf{A}^+ \in \mathbb{ISP}(\mathbf{A})$, by Theorem 3.1, p. 16 of [4] for every $a, b \in A^+$ such that $a \neq b$, there exists a homomorphism $h : A^+ \longrightarrow A$ such that $h(a) \neq h(b)$. Then $\mathbf{A}^+$ is a subdirect product of subalgebras of **A**. Since $\mathbf{A}^+$ is subdirectly irreducible there exists $\mathbf{B} \in S(\mathbf{A})$ such that $\mathbf{A}^+$ is isomorphic to **B**. Then there exists a one-to-one homomorphism from $\mathbf{A}^+$ to **A**, which is a contradiction, since $\mathbf{A}^+$ is finite and $A^+ - A = \{p\}$. □

Corollary 59. *If* $\mathbf{A} \in \mathbb{H}_f$, *then* $\mathbb{ISP}(\mathbf{A}) \subsetneq \mathbb{H}$ *and* $\mathbb{H}_f \nsubseteq \mathbb{ISP}(\mathbf{A})$.

By the above results we can conclude that the variety $\mathscr{H}$ do not admit a natural duality.

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